Algebra, Analysis, and Applications of the Renormalization Group

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Wilsonian RG method for fermions

Flow parameter: typically an energy scale Λ
(sometimes even the temperature T)

integrate out degrees of freedom with energy $\gtrsim \Lambda$

kinetic energy $\gtrsim \Lambda$, in fact

$\rightarrow$ weak-coupling method.

Typical model: Hubbard $U, t, t', \ldots$
$T = (t_{xy})_{xy}$

Why use RG method at all?

Infrared cutoff Λ $\Longrightarrow$ perturbation theory converges

but nonuniformly in Λ

study dependence on Λ.

Algebra *

$$e^{-W(\bar\eta,\eta)} = \int D\bar\psi D\psi \; e^{\overbrace{-(\bar\psi, Q\psi) - V(\bar\psi,\psi)}^{-S(\bar\psi,\psi)} + (\bar\eta,\psi) + (\eta,\bar\psi)}$$

usual functional integral.

Notation $\quad (\bar\eta, \psi) = \sum_\alpha \int_0^\beta d\tau \sum_x \bar\eta_\alpha(\tau,x)\,\psi(\tau,x)$ etc.

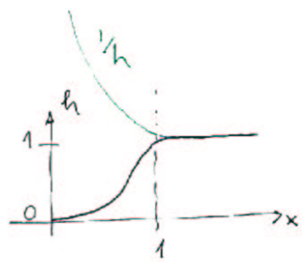
$$(\bar\psi, Q\psi) = T\sum_n \int dp \sum_\alpha \bar\psi_\alpha(\omega_n,p)\,\underbrace{(i\omega_n - e(p))}_{Q(\underline{p})}\,\psi_\alpha(\omega_n,p)$$

$$e(p) = \epsilon(p) - \mu .$$

$Q \to Q_\Lambda \; : \; W \to W_\Lambda$

e.g. $Q_\Lambda(\underline{p}) = Q(\underline{p}) \cdot \chi_\Lambda(\underline{p})^{-1}$

$$\chi_\Lambda(p) = h\left(\frac{|Q(\underline{p})|^2}{\Lambda^2}\right)$$



so the bare propagator gets an IR cutoff

$$C_\Lambda(p) = \frac{1}{i\omega_n - e(p)}\,\chi_\Lambda(p).$$

$$\partial_\Lambda e^{-W_\Lambda(\bar\eta,\eta)} = \underbrace{+\left(\frac{\delta}{\delta\eta}, \dot Q_\Lambda \frac{\delta}{\delta\bar\eta}\right)}_{\Delta_{\dot Q_\Lambda}} e^{-W_\Lambda}$$

$$\Rightarrow \dot W_\Lambda(\bar\eta,\eta) = +\Delta_{\dot Q_\Lambda} W_\Lambda - \left(\frac{\delta W_\Lambda}{\delta\eta}, \dot Q_\Lambda \frac{\delta W_\Lambda}{\delta\bar\eta}\right)$$

avoid Q_Λ in favour of C_Λ;

either transform to

$$G_\Lambda(\bar\psi,\psi) = \text{const.} - (\bar\psi, C_\Lambda^{-1}\psi) + W\left((C_\Lambda^T)^{-1}\bar\psi, C_\Lambda^{-1}\psi\right)$$

$$\Rightarrow \dot G_\Lambda = -\Delta_{\dot C_\Lambda} + \left(\frac{\delta G_\Lambda}{\delta\psi}, \dot C_\Lambda \frac{\delta G_\Lambda}{\delta\bar\psi}\right) \qquad \text{(Polchinski)}$$

or Legendre transform to

$$\Gamma(\bar\phi,\phi) = W(\bar\eta,\eta) - (\bar\eta,\phi) - (\bar\phi,\eta) \qquad \phi = \frac{\delta W}{\delta\bar\eta} \quad \bar\phi = -\frac{\delta W}{\delta\eta}$$

$$^{*)}\Rightarrow \dot\Gamma_\Lambda = (\bar\phi, \dot Q_\Lambda \phi) - \mathrm{Tr}\left(\dot Q_\Lambda \left(\frac{\delta^2\Gamma}{\delta\bar\phi\,\delta\phi}\right)^{-1}\right)$$

another useful variant is the Wick ordered function

$$\mathcal{H}_\Lambda = e^{\Delta_{D_\Lambda}} G_\Lambda, \qquad \text{where} \quad D_\Lambda + C_\Lambda = C.$$

* see, e.g., M.S. & C. Honerkamp, Prog. Theo. Phys. 105, 1 (2001)

*) $\partial_\Lambda \Gamma_\Lambda(\Phi) = (\partial_\Lambda W_\Lambda)(\bar\eta(\Phi), \eta(\Phi))$

Expansion in the fields $\Phi = (\bar\phi, \phi)$

$$\Gamma_\Lambda(\Phi) = (\bar\phi, \gamma_{2,\Lambda}\phi) + \sum_{m\ge 2} \Gamma_\Lambda^{(2m)}(\Phi)$$

$\Gamma_\Lambda^{(2m)}$

2m legs.

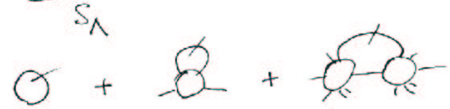
$\gamma_{2,\Lambda}$ = inverse of the full propagator G_Λ

$$\left(\frac{\delta^2\Gamma}{\delta\bar\phi\,\delta\phi}\right)^{-1} = \left(-\gamma_{2,\Lambda} + \frac{\delta^2\Gamma^{(\ge 4)}}{\delta\bar\phi\,\delta\phi}\right)^{-1}$$

$$= -G_\Lambda\left(1 - \frac{\delta^2\Gamma^{(\ge 4)}}{\delta\bar\phi\,\delta\phi}G_\Lambda\right)^{-1}$$

$$\dot\Gamma_\Lambda = (\bar\phi, \dot Q_\Lambda \phi) + \mathrm{Tr}\left(\dot Q_\Lambda G_\Lambda \left(1 - \frac{\delta^2\Gamma^{(\ge 4)}}{\delta\bar\phi\,\delta\phi} G_\Lambda\right)^{-1}\right)$$

$$\mathrm{Tr}\Big(\underbrace{G_\Lambda \dot Q_\Lambda G_\Lambda}_{S_\Lambda}\Big(1 + \frac{\delta^2\Gamma^{(\ge 4)}}{\delta\bar\phi\,\delta\phi}G_\Lambda + \dots\Big)\Big)$$

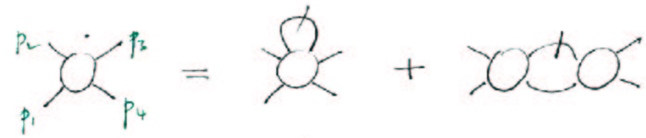


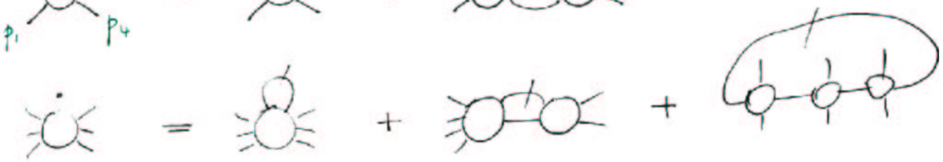
+ . . .

hierarchy of equations $\qquad -S_\Lambda = \frac{Q\cdot\dot\chi_\Lambda}{(Q-\Sigma_\Lambda\chi_\Lambda)^2}$

selfenergy $\dot\Sigma_\Lambda$ =

, $G_\Lambda = \frac{\chi_\Lambda}{Q - \Sigma_\Lambda\chi_\Lambda}$





⋮

Can be used to generate p.th. to all orders as follows: to get order r in V, set $\Gamma^{(2r+2)} = 0$. Solve backwards in m.

The variant based on Polchinski's eq. (or the Wick ordered eq.) gives a very useful representation of the expansion.
(Brydges-Kennedy-formula)

Analysis.

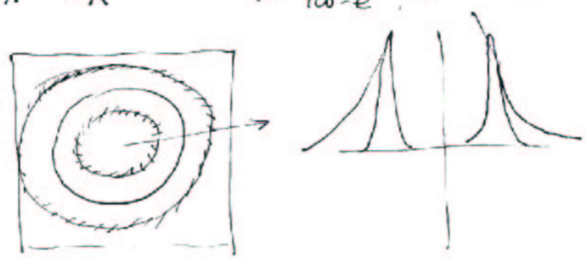
what terms (if any) are small?

truncations of the hierarchy?

1) suppose for now that $\Sigma_\Lambda = 0$. Then

$G_\Lambda = \frac{1}{i\omega - e} \chi_\Lambda = C_\Lambda$ and $S_\Lambda = \frac{1}{i\omega - e} \dot{\chi}_\Lambda = \dot{C}_\Lambda$

supp $\dot{C}_\Lambda$ ⟶



can do power counting for $\Lambda \to 0$ since everything is determined by the singularity of C.

result:

$2m \geq 6$	irrelevant
$2m = 4$	marginal
$2m = 2$	relevant.

less obvious bounds:

$2m = 4$ marginal only ~~at~~ for 1-loop type graphs at particular momenta

if there is no perfect nesting

2) with selfenergy:

deformation of the Fermi surface

We have proven that in absence of perfect nesting and van Hove singularities,

the Fermi surface deformation is regular (no curvature singularities)

and v_F and Z are finite & nonzero everywhere near the F.S.

There is a one-to-one correspondence between ~~free and f~~ the F.S. of the free and the interacting model.

[counterterms vs. no counterterms: that's a matter of taste]

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Part I (Introduction to the Method): see the scanned notes or (e.g.)

M. Salmhofer and C. Honerkamp, Prog. Theo. Phys. 105, 1 (2001)

Applications I: “Easy” Cases

1d Luttinger and Hubbard model (non–half–filled)
Benfatto, Gallavotti, Procacci, Scoppola

2d fermions: idea and implementation of angular sectorization
Feldman, Magnen, Rivasseau, Trubowitz

2d, non–nested Fermi surfaces: study of selfenergy regularity and Fermi surface shift
Feldman, Salmhofer, Trubowitz

Clarification about zero-temperature limit and “anomalous diagrams”
Feldman, Knörrer, Salmhofer, Trubowitz

2d jellium: proof of Fermi liquid behaviour above $T_c \sim e^{-c/|U|}$
Disertori, Rivasseau

2d non–time–reversal–symmetric: Fermi liquid at $T = 0$
Feldman, Knörrer, Trubowitz

all this is weak–coupling

Applications II: “Hard” Cases

Hubbard models U, t, t'; 3 main effects in interesting density range

nesting
van Hove singularities
Umklapp scattering

why “hard”? Flow always leads to strong coupling, hence has to be stopped at a nonzero scale Λ_*.

C. Honerkamp, T.M. Rice, N. Furukawa

C. Halboth, W. Metzner

D. Zanchi, H. Schulz

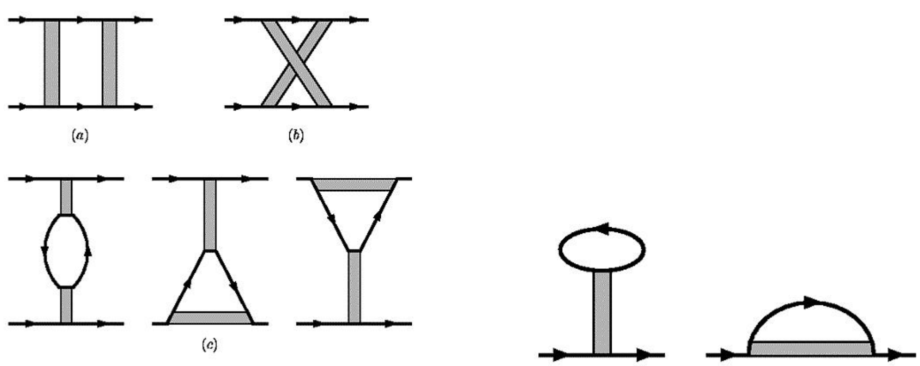
B. Binz, B. Douçot, D. Baeriswyl

C. Bourbonnais

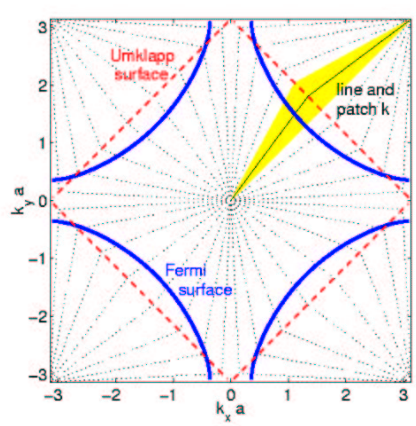
T. Busche, L. Bartosch, P. Kopietz

Spin symmetry

$$F^{(\Lambda)}_{\alpha_1,\ldots,\alpha_4}(p_1,p_2,p_3) = V_\Lambda(p_1,p_2,p_3)\delta_{\alpha_1,\alpha_4}\delta_{\alpha_2,\alpha_3} - V_\Lambda(p_2,p_1,p_3)\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4}$$



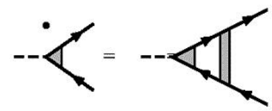
Projection to the Fermi surface



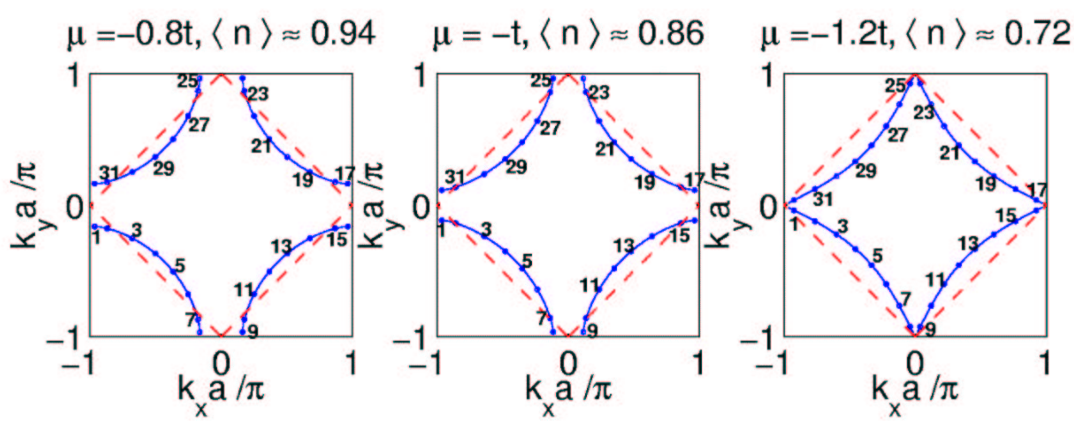
Project to $\omega_i = 0$ and p_i on the Fermi surface $\rightarrow V_\Lambda(\theta_1,\theta_2,\theta_3)$

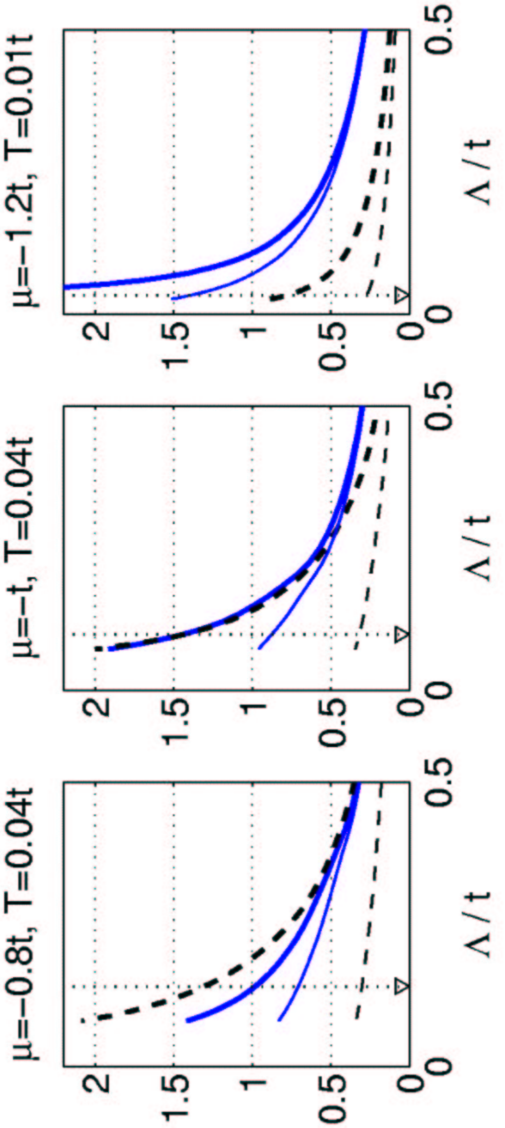
Response to external fields

d-wave pairing	$\Phi_{s,d-\mathrm{sc}} \sum_{\vec{k}} h_{d-\mathrm{sc}}(\vec{k})\, c^\dagger_{\vec{k},s} c^\dagger_{-\vec{k},-s}$.
spin-density wave	$\Phi_{z,\mathrm{AF}} \sum_{\vec{k}} h_{\mathrm{AF}}(\vec{k}) \left(c^\dagger_{\vec{k}+(\pi,\pi),s} c_{\vec{k},s} - c^\dagger_{\vec{k}+(\pi,\pi),-s} c_{\vec{k},-s} \right)$
d–wave Pomeranchuk	$\lim_{\vec{q}\to 0} \Phi_{d-\mathrm{P}} \sum_{\vec{k},s} h_{d-\mathrm{P}}(\vec{k})\, c^\dagger_{\vec{k}+\vec{q},s} c^\dagger_{\vec{k},s}$,
d-density wave	$\Phi_{d-\mathrm{DW}} \sum_{\vec{k},s} h_{d-\mathrm{dw}}(\vec{k})\, c^\dagger_{\vec{k}+(\pi,\pi),s} c_{\vec{k},s}$

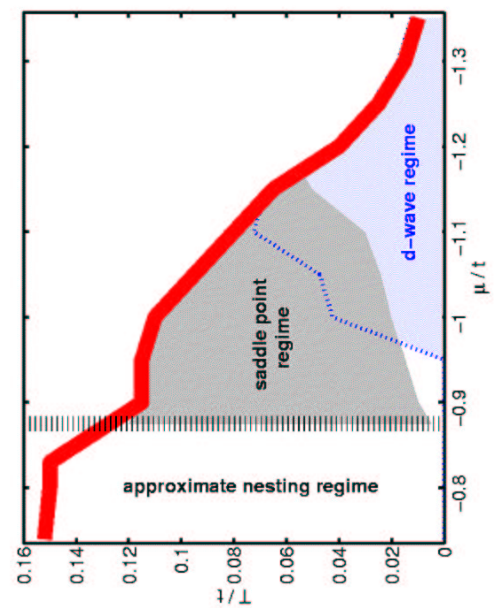


$U = 3t,\ t'/t = -0.3$: Three Regimes

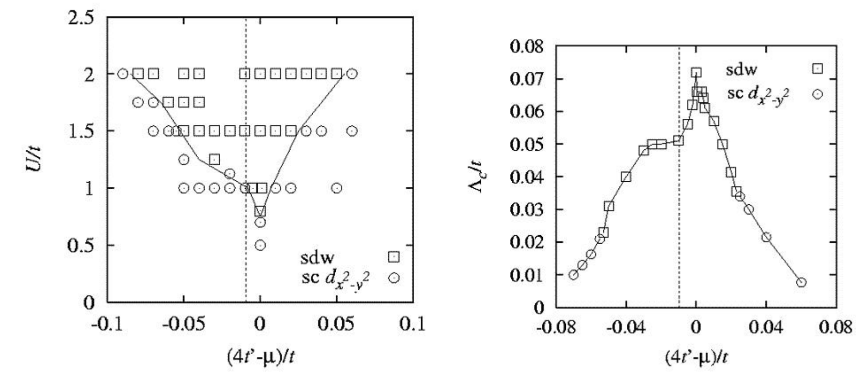




d–wave susceptibility vs. antiferromagnetic susceptibility

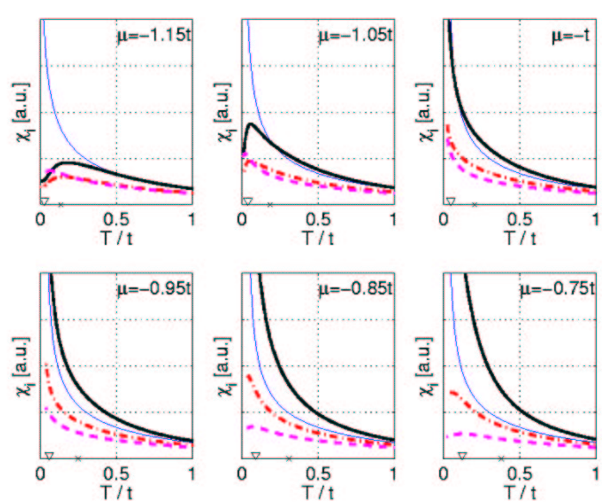


C.Halboth and W.Metzner, PRL 85, 5162 (2000)



Possibility of a Pomeranchuk instability

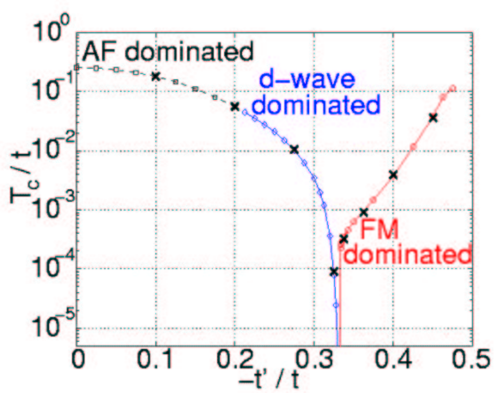
C. Honerkamp, M. Salmhofer, T.M. Rice, EPJ B (2002)



A. Neumayr and W. Metzner, cond-mat/0208431:

renormalized perturbation theory with superconductivity and Pomeranchuk instability: breaking of the lattice rotation symmetry occurs for a density range

Where is the Ferromagnetism?



What does this “flow to strong coupling” mean?

Approximations justified beyond weak coupling range (overlapping loop effects) if no perfect nesting

flows with initially small gaps: gap increases, couplings do not diverge.

What about the selfenergy?

Fermi surface flow studied by Honerkamp et al., PR B 63, 035109 (2001): deformation is small.

Z factor: studied by Zanchi (Europhys. Lett. 55, 376 (2001)) for the half–filled case with $t' = 0$.

by Honerkamp & M.S. for general fillings (in preparation)

full selfenergy effects remain to be investigated.