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V. Toledano Laredo

Stability conditions and Stokes factors

Physics $\Rightarrow$ solution of WCF (Gaiotto-Moore-Neitzke)

Mathematics $\Rightarrow$ solution of WCF (Reineke, Joyce, Kontsevich-Soibelman)

$\leftrightsquigarrow$ isomonodromic deformation (TL + Bridgeland)
Frobenius manifold structures

$\mathcal{A}$ = abelian category branes (~~anti~~branes)

$$0 \to S_2 \to S \to S_1 \to 0$$

$$\updownarrow$$

Ex

1) $\mathcal{A} = \mathrm{Rep}(\overset{1}{\bullet} \to \overset{2}{\bullet})$ indecomposables $\underset{S_1}{\mathbb{C} \to 0}$ $\underset{S_2}{0 \to \mathbb{C}}$ $\underset{S}{\mathbb{C} \xrightarrow{1} \mathbb{C}}$

2) $\mathcal{A} = \mathrm{Coh}(X)$, X smooth projective$/\mathbb{C}$ B-branes.

Stability conditions on $\mathcal{A}$

central charge

$$Z : K(\mathcal{A}) \longrightarrow \mathbb{C}$$

$$\begin{array}{c} \cup \\ [M] \longrightarrow \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\} \sqcup \mathbb{R}_{<0} \\ \cap \\ K_{>0}(\mathcal{A}) \end{array}$$

(2)

$K(\mathcal{A})$ Grothendieck sp changes

$K(\cdot \to \cdot) = \mathbb{Z}^2$

$K_{num}(Coh(X)) \cong \mathbb{Z}^2$

X alg. curve

$\xi \longrightarrow (\deg \xi, \operatorname{rank} \xi)$

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eg for stability conditions

i) $Rep(\cdot \to \cdot)$ $Z(S_i) = z_i \in \mathbb{H}$

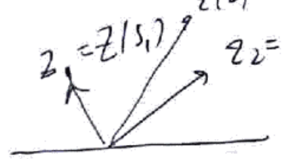
ii) $[\xi] \longrightarrow -\deg \xi + i \operatorname{rank} \xi$

Semistable objects BPS branes.

$M \in \mathcal{A}$, $\phi(M) := \frac{1}{\pi} \arg Z(M) \in (0,1]$ phase

Def M is semistable (wrt Z) if $N \subseteq M \Rightarrow \phi(N) \leq \phi(M)$

eg i) $Rep(\cdot \to \cdot)$

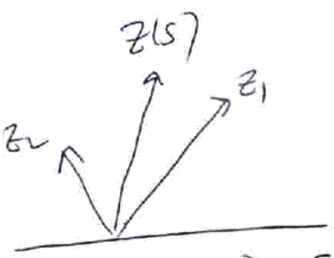


$S = \overset{\mathbb{C}}{\bullet} \xrightarrow{1} \overset{\mathbb{C}}{\bullet}$

$S_2 \subseteq S$ is the only subobject $\Rightarrow S$ is semistable.

BPS branes S_1, S_2, S and their multiples.

<u>Lemma</u> $M \oplus N$ semistable
$\Leftrightarrow$ M, N are semistable
<u>and</u> $\phi(M) = \phi(N)$.



$\phi(S_2) > \phi(S) \Rightarrow S$ <u>not</u> semistable

BPS branes are S_1, S_2 (and multiples)

ii) $\mathcal{A} = Coh(X)$, $\dim X = 1$.

BPS branes are semistable vector bundles and torsion sheaves

<u>Goal</u>: (i) "Count" semistable objects of charge $\alpha \in K(\mathcal{A})$

(ii) Show WCF.

$$\mathcal{X}_\alpha(Z) = \{ M \in \mathcal{A} \mid M \text{ semistable}, [M] = \alpha \in K \}$$

$\Big\}$ "Count"

$\Omega(\alpha; Z)$ "DT inv."

eg $\Omega(\alpha, Z) = |\mathcal{X}_\alpha(Z)|$ if $\mathcal{A}/\mathbb{F}_q$

$\Omega(\alpha, Z) = \chi(\mathcal{X}_\alpha(Z))$

Basic observation (Reinecke, Joyce, KS)

1. It is MUCH simpler to control WC for $\mathcal{X}_\alpha(z)$ (rather that $\Omega(\alpha, z)$).
2. $\mathcal{X}_\alpha(z) \rightsquigarrow$ Hall algebra of $\mathcal{A}$.

$$\begin{array}{ccc} \} & & \downarrow \text{"}\int\text{"} \\ \downarrow & & \\ \Omega(\alpha, z) & & \text{torus of } K(\mathcal{A}). \end{array}$$

Hall alg: $H_{\mathcal{A}}$ = "functions" on moduli space $\mathcal{M}$ of objects of $\mathcal{A}$.

*) $\mathcal{M}$ is a stack

*) functions = constructible functions

$$= \left\{ \sum a_i \delta_{Y_i} \;\middle|\; Y_i \subseteq \mathcal{M} \text{ locally closed sub stacks} \right\}$$

torus $T = K(\mathcal{A})^\vee \otimes \mathbb{C}^\times$

$$\alpha \in K \to e^\alpha \in \mathbb{C}[T]$$

Poisson str. on T:

$$\{e^\alpha, e^\beta\} = \langle \alpha, \beta \rangle e^{\alpha+\beta}$$

$$(\alpha, \beta) = \sum_i (-1)^i \dim \mathrm{Ext}^i(\alpha, \beta) \qquad \text{Euler form}$$

$$\langle \alpha, \beta \rangle = \tfrac{1}{2}\big((\alpha, \beta) - (\beta, \alpha)\big)$$

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Conj $\int : \mathcal{H}(\mathcal{A}) \to$ Diff ops on T

$$\begin{array}{c} \cup \\ \text{Vector fields} \\ \uparrow \\ (\mathbb{C}[T], \{,\}) \end{array}$$

? $\int$ exists only in VERY favorable circumstances

1) gldim $(\mathcal{A}) \leq 1$ (Rep(quivers), $Coh(X)$, $\dim X = 1$)
Reineke

or 2) $\mathcal{A}$ CY3 (KS)

If $\dim \mathcal{A} \leq 1$:

$$\mathcal{X}_\alpha(z) \rightsquigarrow S_\alpha^{(z)} \in \mathcal{H}(\mathcal{A})$$

$$\Big\downarrow \int$$

$$\Omega(\alpha, z)\{e^\alpha, -\}$$

<u>Algebra structure on $\mathcal{H}(\mathcal{A})$</u>

$$f * g(M) = \int_{N \subseteq M} f(N) \cdot g(M/N) \quad \left(= \sum_{N \subseteq M} f(N) g(M/N) \quad \text{if } \mathcal{A}/\mathbb{F}_q \right)$$

$\int$ = pushforward of constructible fns: $\int \delta_{Y_i} = \chi(Y_i)$.

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$H(\mathcal{A}), *$ is morally related to Harvey-Moore algebra of BPS states.
(indept of Z) (dep on Z)

Unit: $\quad 1_0(M) = \begin{cases} 1 & \text{if } M=0 \\ 0 & \text{else.} \end{cases}$

$$\neq$$

$$\mathbb{1}(M) = \{ 1 \quad \forall M.$$

$$f_1 * \cdots * f_n(M) = \sum_{M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M} f_1(M_1) f_2(M_2/M_1) \cdots f_n(M_n/M_{n-1})$$

WCF $\Leftarrow$ Harder Narasimhan filtrations.

$E \in \mathcal{A}$, $\exists!$ $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$

s.t. ① $F_i = E_i / E_{i-1}$

② $\phi(F_i) > \phi(F_{i+1})$

Reformulation (Reinecke)

$$\ell \subset \mathbb{H} \text{ ray } \rightsquigarrow SS_\ell = 1_0 + \sum_{\alpha : Z(\alpha) \in \ell} \delta_\alpha$$

char. fn. of all BPS branes of charge in ℓ.

$$HN \Rightarrow \overset{\curvearrowright}{\prod_{\ell \subset \mathbb{H}}} SS_\ell = \mathbb{1} \in \widehat{\mathfrak{H}}$$

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Cor
$$\prod_{\ell \in H}^{\curvearrowright} SS_\ell^{Z} = \prod_{\ell \subset H}^{\curvearrowright} SS_\ell^{Z'} \quad \forall Z, Z'$$
$$\left(\text{"} \mathbb{1} \text{"} \right)$$

WC in $\mathcal{H}_{\mathcal{A}}$.

WCF in Symp(T)

Prop
$$\int SS_\ell^{Z} = \prod_{\substack{\alpha : Z(\alpha) \in \ell \\ \alpha \text{ indivisible}}} K_\alpha^{\Omega(\alpha, Z)}$$

Frobenius manifolds ← Gaiotto - Moore - Neitzke

WCF ~~~> HK metrics (tt^*)

Joyce's work (Bridgeland): Frobenius manifold structure on $\overset{M}{\overset{\shortparallel}{\mathrm{Stab}(\mathcal{A})}}$?

$\Downarrow$

flat connection on $M \times \mathbb{P}^1 \ni t$

with regular sing at $t = \infty$

irregular at $t = 0$.

(8)

Joyce's gen. function:

$$f_\alpha(Z) = \sum_{n\geq 1} \sum_{\alpha_1+\dots+\alpha_n=\alpha} J_n(Z(\alpha_1),\dots,Z(\alpha_n))\, \delta_{\alpha_1} * \dots * \delta_{\alpha_n}$$

$$J_n : (\mathbb{C}^\times)^n \to \mathbb{C}$$ "GW potential"

Thm (Joyce)

① $\exists\ J_n$ s.t. $f_\alpha : \mathrm{Stab}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})_\alpha$ are cts & holomorphic

② the J_n are (essentially) unique & indept of $\mathcal{A}$.

③ $$d f_\alpha = \frac{1}{2} \sum_{\beta+\gamma=\alpha} [f_\beta, f_\gamma]\, d\log\left(\frac{\beta}{\gamma}\right)$$

Bridgeland + TL:

$$f = \sum_\alpha f_\alpha, \quad f_\alpha \in (\mathcal{H}_\mathcal{A})_\alpha$$

$$\nabla_{\mathcal{A},Z} := d - \left(\frac{Z}{t^2} + \frac{f}{t}\right)dt,$$ connection on $\mathbb{P}^1$ with values in $\mathcal{H}_\mathcal{A} \rtimes \mathfrak{h}$

$$\mathfrak{h} = K(\mathcal{A})^\vee \otimes \mathbb{C}.$$

Thm (B-TL) The following are equivalent:

(i) $\bar{f}_\alpha = f_\alpha$ is given by Joyce's gen. fn.

(ii) The Stokes factor of $\nabla_{A,Z}$ corr. to a ray ℓ is SS_ℓ

(iii) The Stokes matrices of $\nabla_{A,Z}$ are $\underset{\text{upper half plane}}{S_+ = \mathbb{1}}$, $\underset{\text{lower half plane}}{S_- = 1_0}$

<u>Cor</u> $\nabla_{A,Z}$ is an isomonodromic family of connection on $\mathbb{P}^1$.

Riemann–Roch / $\mathbb{F}_q$

$$\int \delta_{[M]} = \frac{e^M}{|\mathrm{Aut}\, M|}$$

$$\int \tilde{\delta}_{[M]} = \chi([M])\, \{e^M, -\}/\mathbb{C} \quad \text{in } \dim = 1 \ !!$$

$$q = [\mathbb{L}]$$