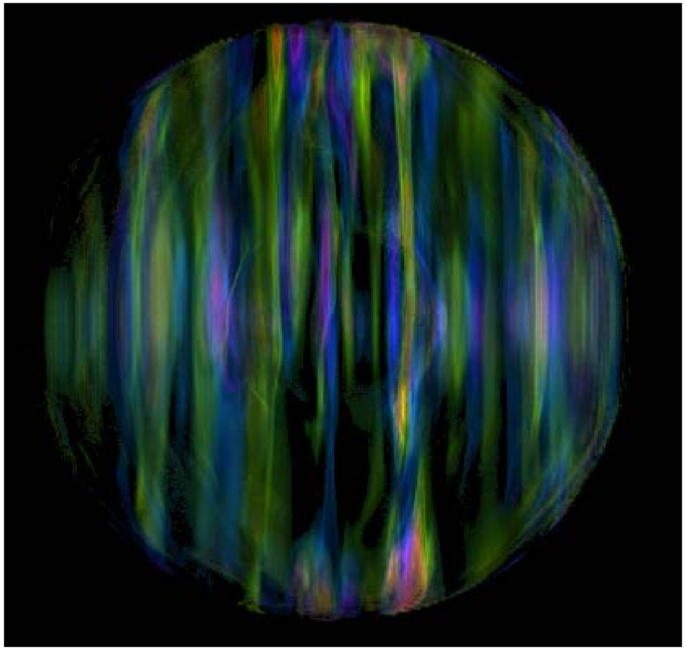
Sub-grid-scale models for dynamo simulations

Bruce Buffett and Hiroaki Matsui

University of California



(vertical vorticity)

KITP Workshop, July 14, 2008

The Problem

Numerical simulations cannot resolve the vast range of scales

Large Eddy Simulations (LES)

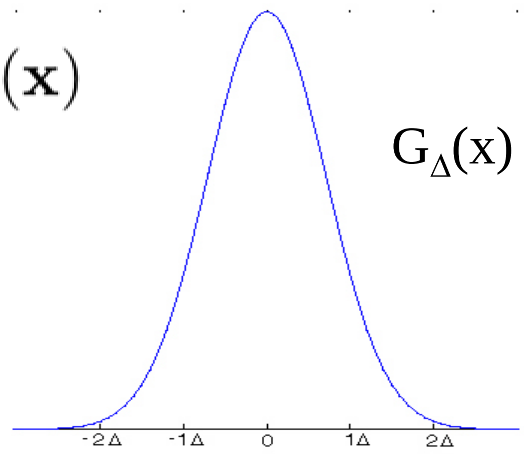
apply spatial filter to eliminate scales smaller than the grid spacing Δ

$\Rightarrow$ introduces additional terms in the governing equations

Spatial Filtering

Eliminate scales smaller than grid spacing Δ

$$\widetilde{V_i(x)} = \int V_i(\mathbf{x}')G_\Delta(\mathbf{x}-\mathbf{x}')\,d\mathbf{x}' = G_\Delta * V(\mathbf{x})$$



Apply filter to heat equation

$$\frac{\partial \widetilde{T}}{\partial t} = -\widetilde{\nabla \cdot (\mathbf{V}T)} + \kappa\widetilde{\nabla^2 T}$$

Express in terms of filtered (e.g. resolved) quantities

Resolved Fields

1. Interchange order of filtering and spatial differentiation

$$\frac{\partial \tilde{T}}{\partial t} = -\nabla \cdot (\widetilde{\mathbf{V}T}) + \kappa \nabla^2 \tilde{T} - \mathcal{D}_1(\mathbf{V}T) + \kappa\, \mathcal{D}_2(T)$$

where

$$\mathcal{D}_1(\mathbf{V}T) = \widetilde{\nabla \cdot (\mathbf{V}T)} - \nabla \cdot (\widetilde{\mathbf{V}T})$$

$$\mathcal{D}_2(T) = \widetilde{\nabla^2 T} - \nabla^2 \tilde{T}$$

are corrections for the commutation error

Resolved Fields

2. Express in terms of resolved (filtered) quantities

$$\frac{\partial \widetilde{T}}{\partial t} = -\nabla \cdot (\widetilde{\mathbf{V}}\widetilde{T}) + \kappa \nabla^2 \widetilde{T} - \nabla \cdot \mathbf{I} - \mathcal{D}_1(\mathbf{V}T) + \kappa \mathcal{D}_2(T)$$

where

$$\mathbf{I} = \widetilde{\mathbf{V}T} - \widetilde{\mathbf{V}}\,\widetilde{T}$$

is the sub-grid scale (SGS) heat flux

Sub-Grid-Scale Terms

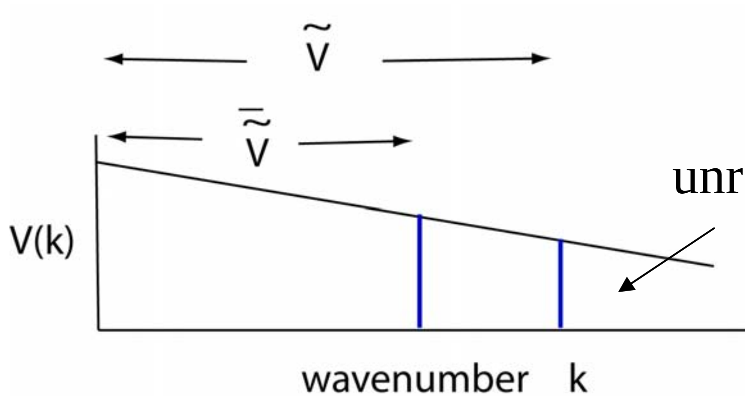
Four SGS terms in dynamo model

1. $I_i = \widetilde{V_i T} - \widetilde{V}_i \widetilde{T}$ SGS heat flux
2. $T_{ij} = \widetilde{V_i V_j} - \widetilde{V}_i \widetilde{V}_j$ SGS momentum flux
3. $M_{ij} = \widetilde{B_i B_j} - \widetilde{B}_i \widetilde{B}_j$ SGS Maxwell stress
4. $a_i = \epsilon_{ijk} (\widetilde{V_j B_k} - \widetilde{V}_j \widetilde{B}_k)$ SGS Induction

Scale-Similarity Model

Apply a test filter to numerical (resolved) solution

$$\overline{\widetilde{\mathbf{V}}} = G_{2\Delta} * \widetilde{\mathbf{V}}$$



unresolved scales

Approximate SGS term

$$I_i = \widetilde{V_i T} - \widetilde{V}_i \widetilde{T} \quad \Longrightarrow \quad I_i^{sim} = C^{sim}\left(\overline{\widetilde{V}_i \widetilde{T}} - \bar{\widetilde{V}}_i \bar{\widetilde{T}}\right)$$

Other SGS Terms

Scale similarity can be used for other SGS terms in dynamo problem

1. $I_i^{sim} = C_I^{sim}(\overline{\tilde{V}_i \tilde{T}} - \overline{\tilde{V}_i}\,\overline{\tilde{T}})$ SGS heat flux

2. $T_{ij}^{sim} = C_T^{sim}(\overline{\tilde{V}_i \tilde{V}_j} - \overline{\tilde{V}_i}\,\overline{\tilde{V}_j})$ SGS momentum flux

3. $M_{ij}^{sim} = C_M^{sim}(\overline{\tilde{B}_i \tilde{B}_j} - \overline{\tilde{B}_i}\,\overline{\tilde{B}_j})$ SGS Maxwell stress

4. $a_i^{sim} = C_a^{sim}\epsilon_{ijk}(\overline{\tilde{V}_j \tilde{B}_k} - \overline{\tilde{V}_j}\,\overline{\tilde{B}_k})$ SGS Induction

Note on Implementation

Implement the scale-similarity model

$$I_i^{sim} = C^{sim}(\overline{\widetilde{V}_i\widetilde{T}} - \bar{\widetilde{V}}_i\bar{\widetilde{T}})$$

by expanding fields $\widetilde{\mathbf{V}}$ and $\widetilde{T}$ in Taylor series

$$\widetilde{\mathbf{V}}(\mathbf{x}) \approx \widetilde{\mathbf{V}}(\mathbf{x}_0) + \nabla\widetilde{\mathbf{V}} \cdot (\mathbf{x} - \mathbf{x}_0)$$

$$\widetilde{T}(\mathbf{x}) \approx \widetilde{T}(\mathbf{x}_0) + \nabla\widetilde{T} \cdot (\mathbf{x} - \mathbf{x}_0)$$

After a little algebra

$$I_i^{sim} \approx C^{sim}\gamma_2 \frac{\partial\widetilde{V}_i}{\partial x_k}\frac{\partial\widetilde{T}}{\partial x_k}$$

(Leonard, 1974)

Commutation Error

Error due to interchanging filter operator and spatial derivative

$$\mathcal{D}_i(f) \equiv \widetilde{\frac{\partial f}{\partial x_i}} - \frac{\partial \tilde{f}}{\partial x_i} \approx -\frac{1}{2}\left(\frac{\partial \gamma_2}{\partial x_i}\right)\frac{\partial^2 f}{\partial x_k \partial x_k}$$

$$\approx -\frac{C^{diff}}{2}\left(\frac{\partial \Delta^2}{\partial x_i}\right)\frac{\partial^2 \tilde{f}}{\partial x_k \partial x_k}$$

commutation error vanishes if filter width Δ is constant

Germano Identity

Define $$I_i^{\Delta} = \widetilde{V_i T} - \tilde{V}_i \tilde{T}$$

$$I_i^{2\Delta} = \overline{\widetilde{V_i T}} - \overline{\tilde{V}_i}\,\overline{\tilde{T}}$$

$$\overline{I_i^{\Delta}} = \overline{\widetilde{V_i T}} - \overline{\tilde{V}_i \tilde{T}}$$

Germano identity

$$I_i^{2\Delta} - \overline{I_i^{\Delta}} = \overline{\tilde{V}_i \tilde{T}} - \overline{\tilde{V}_i}\,\overline{\tilde{T}}$$

Use model to evaluate $I_i^{2\Delta}$ and $\overline{I_i^{\Delta}}$

Dynamic Scale Similarity Model

Germano identity $\quad I_i^S = I_i^{2\Delta} - \overline{I_i^{\Delta}}$

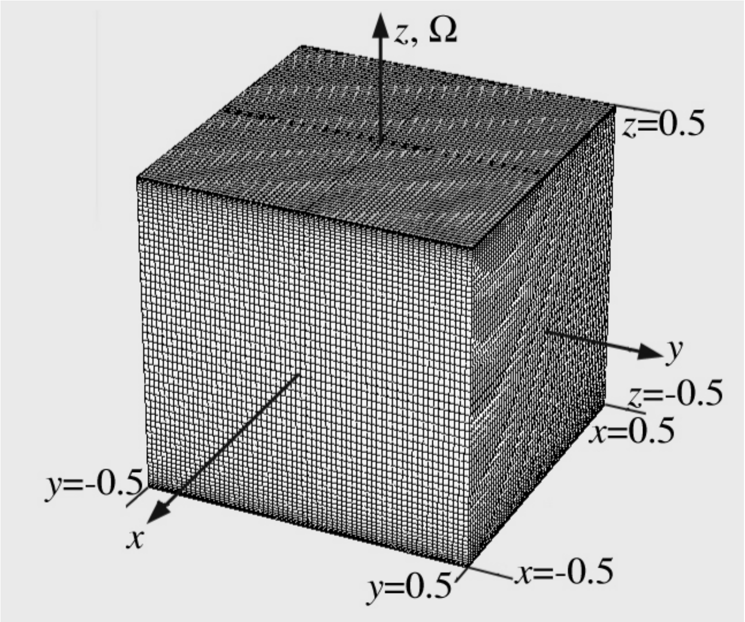
$$= c\left(\gamma^{2\Delta}\frac{\partial \tilde{V}_i}{\partial x_k}\frac{\partial \tilde{T}}{\partial x_k} - \gamma^{\Delta}\overline{\frac{\partial \tilde{V}_i}{\partial x_k}\frac{\partial \tilde{T}}{\partial x_k}}\right)$$

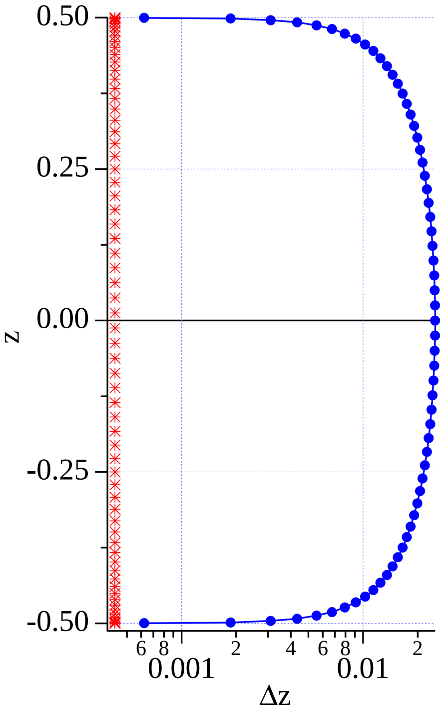
$$\underbrace{\qquad\qquad}_{M_i}$$

Solve for constant $\quad c = \dfrac{I_i^S M_i}{M_i M_i}$

Dynamic model $\quad I_i^{\Delta} = c\,\gamma^{\Delta}\dfrac{\partial \tilde{V}_i}{\partial x_k}\dfrac{\partial \tilde{T}}{\partial x_k}$

Plane-Layer Dynamo



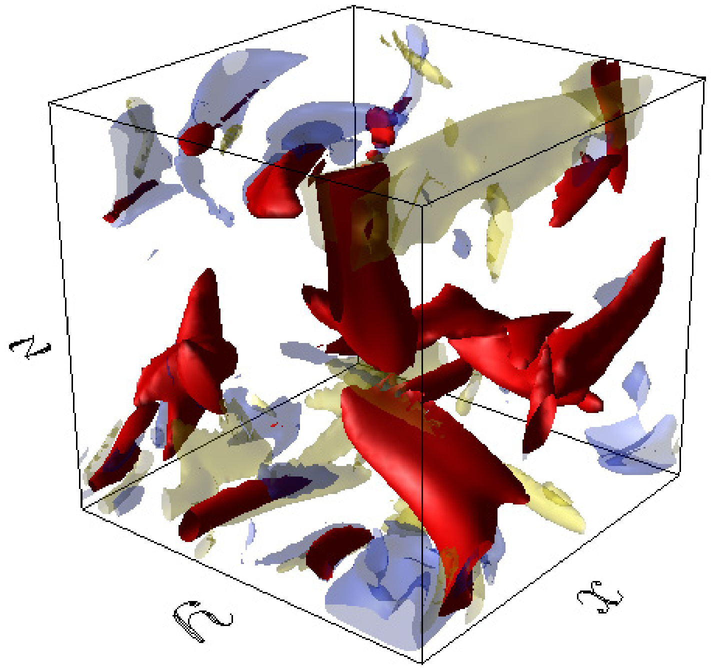


Dimensionless parameters: Pr = 1, Pm = 1, E = 4×10^{-4}, Ra = 10^3

Test SGS Models

Intensity of magnetic field |B|

Resolution: 64^3 grid



Approach

i) filter 64^3 solution on to 32^3 grid

ii) evaluate SGS models using 32^3 solution

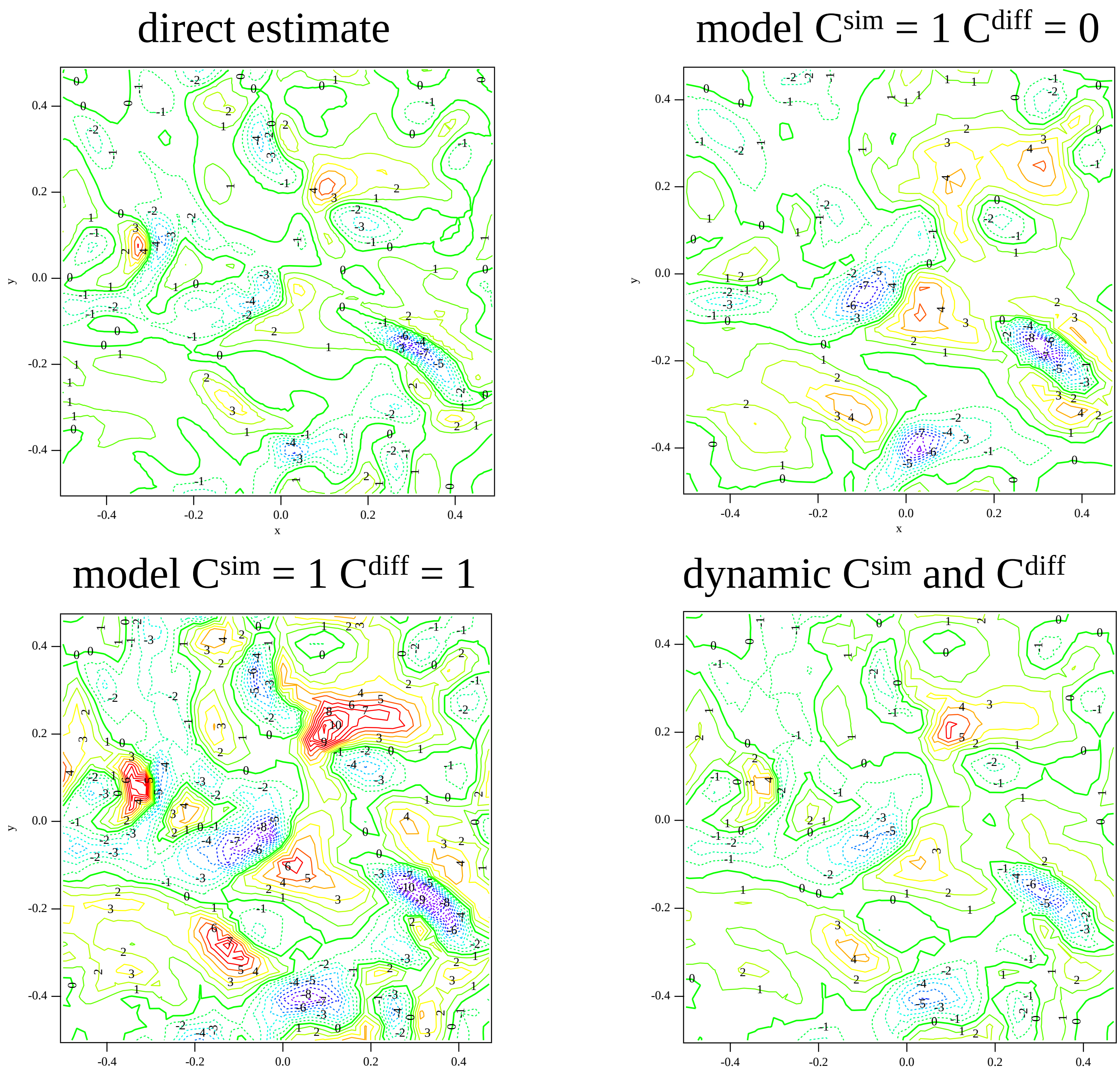
iii) compare with true SGS estimate based on 64^3 solution

SGS Heat Flux

$-\nabla \cdot \mathbf{I}$

Surface

z = -0.176



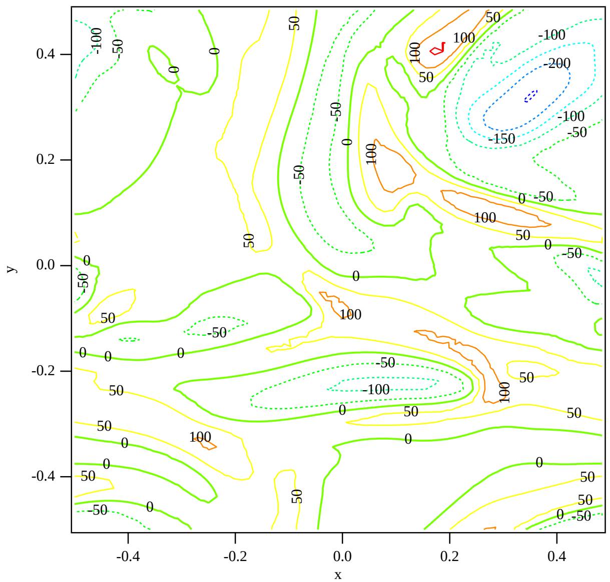
SGS Heat Flux

$-\nabla \cdot \mathbf{I}$

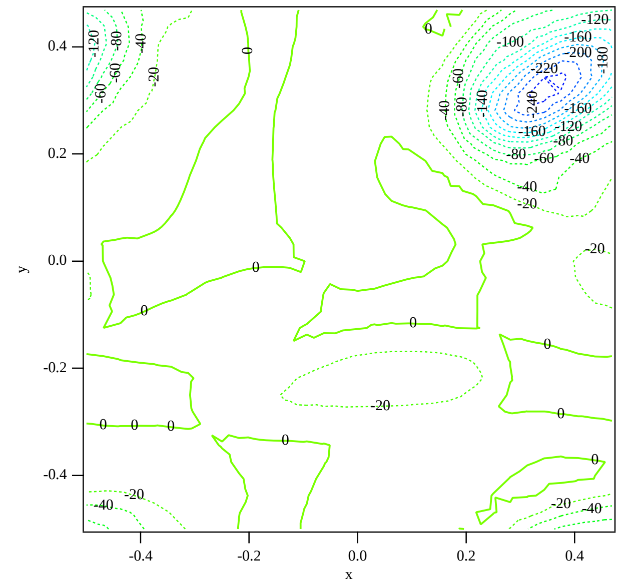
Surface

$z = -0.4975$

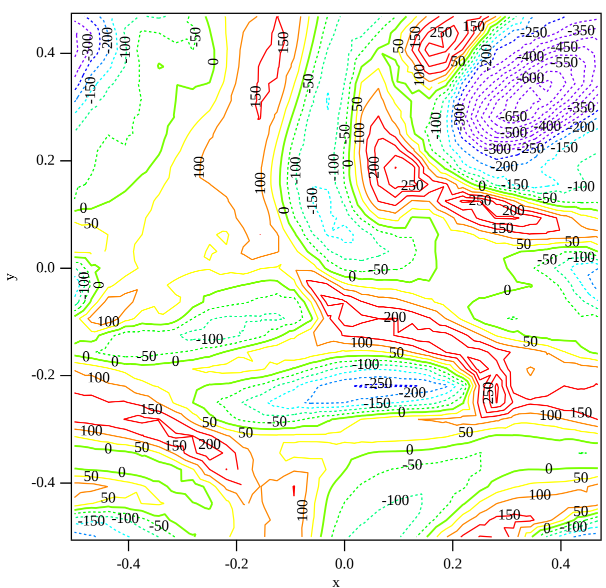
direct estimate



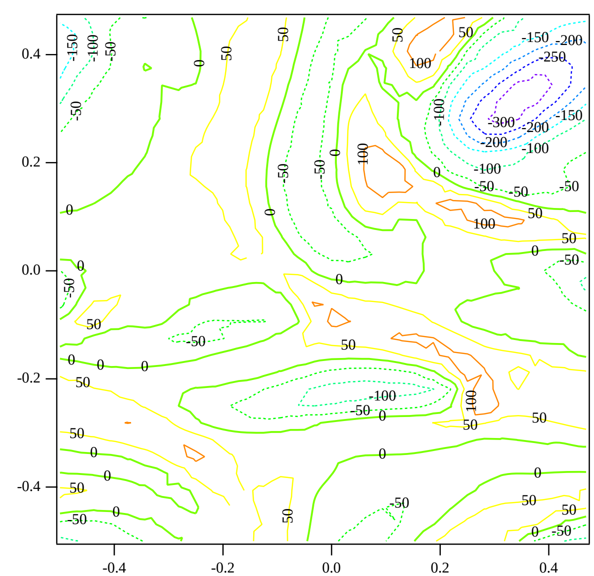
model $C^{sim} = 1$ $C^{diff} = 0$



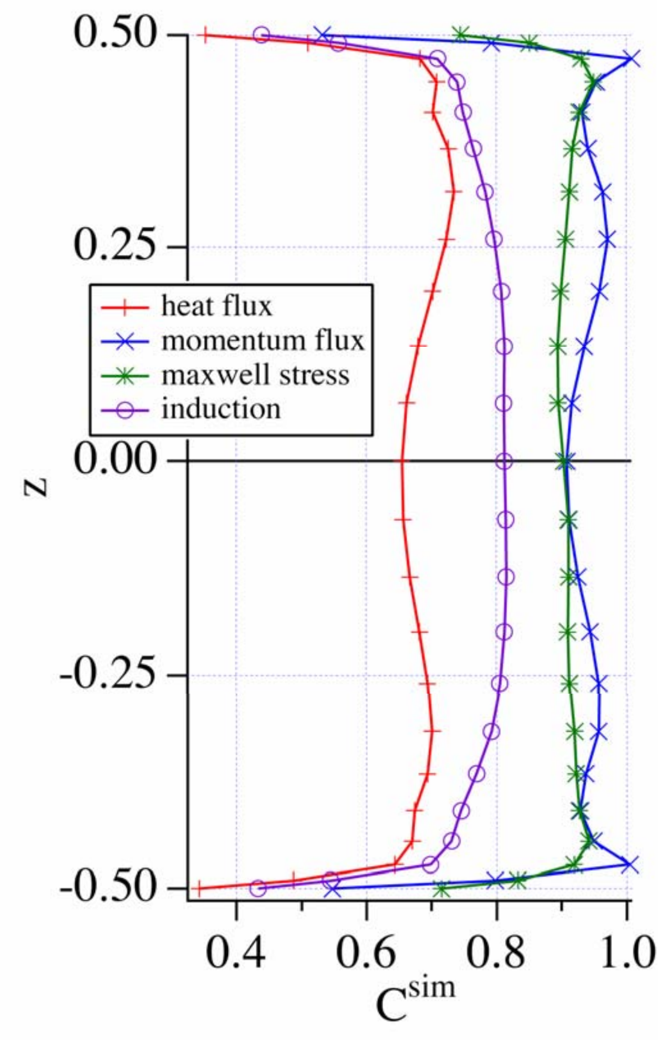
model $C^{sim} = 1$ $C^{diff} = 1$



dynamic C^{sim} and C^{diff}

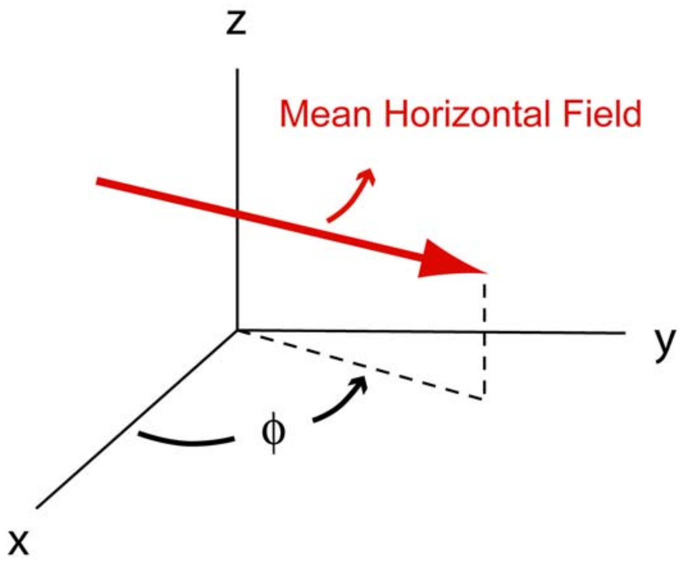


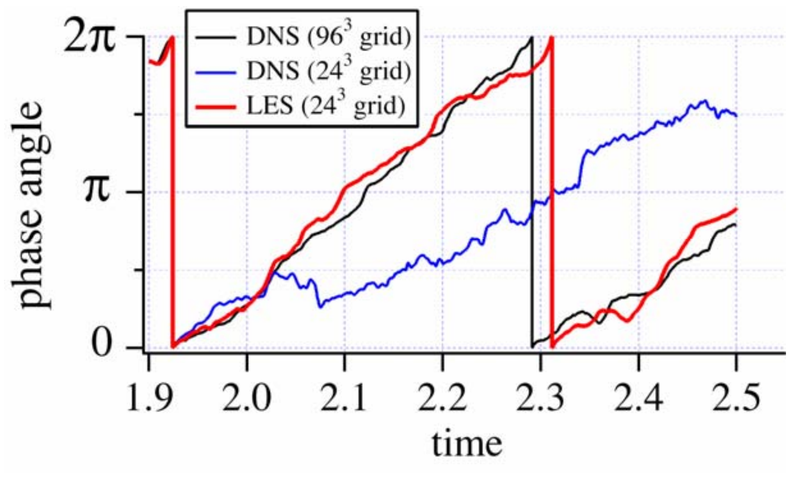
Model Coefficients



Large Eddy Simulations

Evolution of large-scale structure

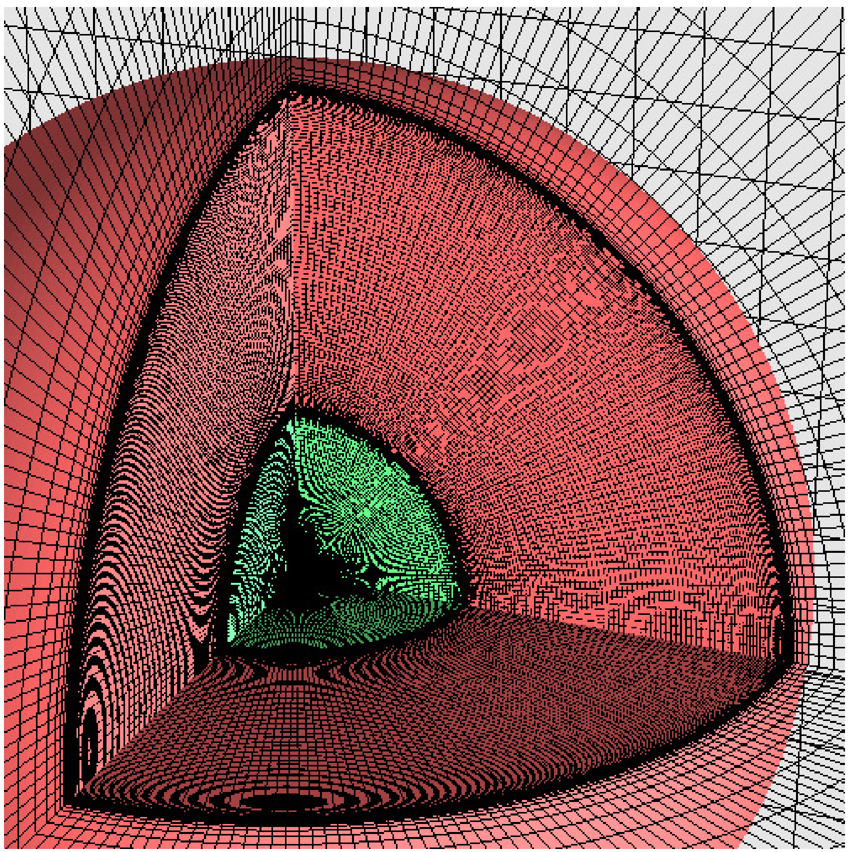




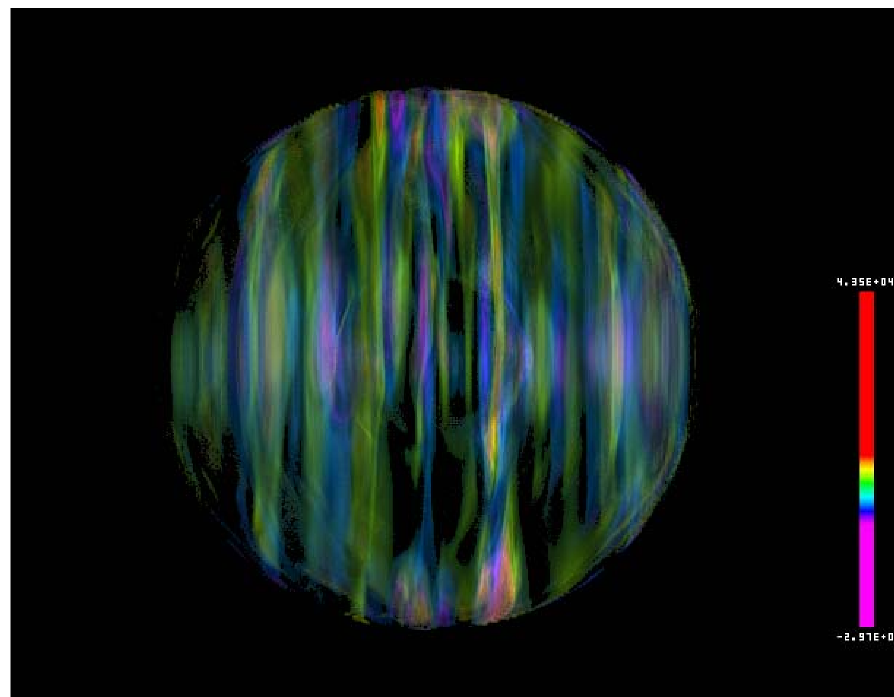
relative number of grid points $96^3 / 24^3 = 64$

Adapt to Spherical Geometry

Spherical Grid



Vertical Vorticity ω_z



Filtering Spatial Derivatives

$$\overline{\frac{\partial f}{\partial x}} = \int \frac{\partial f}{\partial x'} G_\Delta(x - x')\, dx'$$

$$= -\int f(x') \frac{\partial G_\Delta(x - x')}{\partial x'}\, dx'$$

$$= \int f(x') \frac{\partial G_\Delta(x - x')}{\partial x}\, dx'$$

$$= \frac{\partial}{\partial x} \int f(x') G_\Delta(x - x')\, dx' = \frac{\partial \overline{f}}{\partial x}$$

Broader Objectives

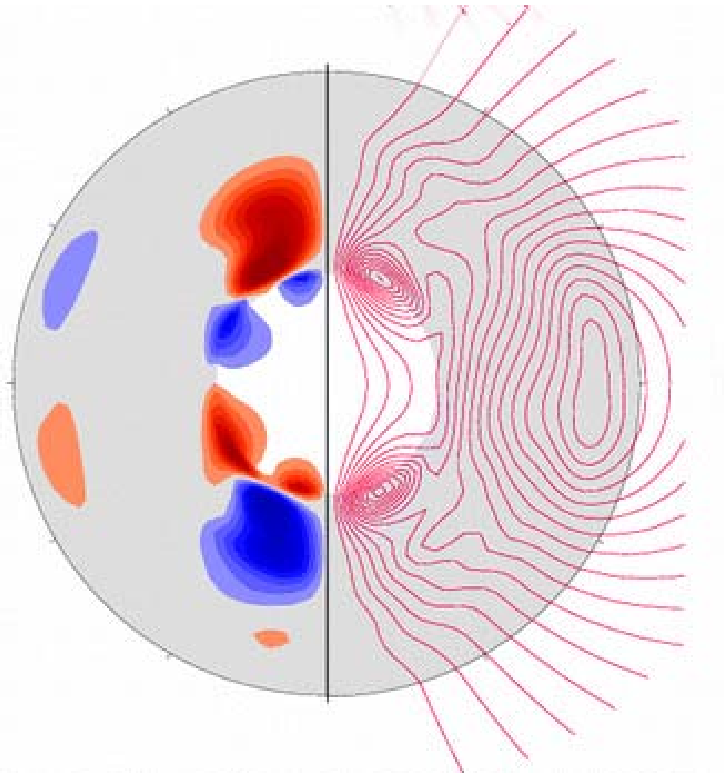
Develop a 2^{nd} generation dynamo model

i) more sophisticated turbulence models

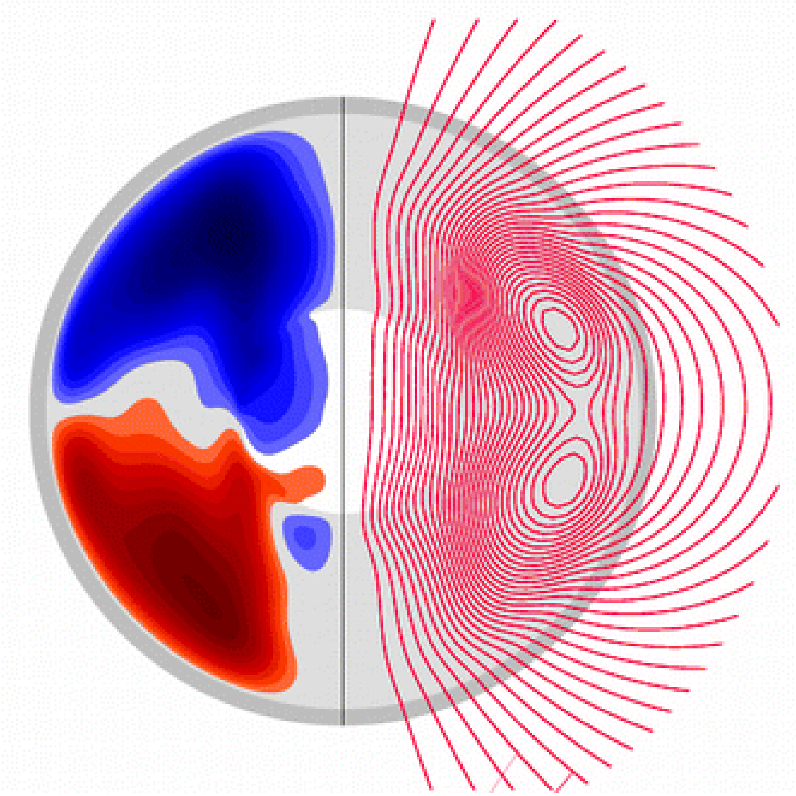
ii) more efficient use of massively parallel computing

⇨ Revisit important problems with more reliable dynamo models

Compensate for High Viscosity?

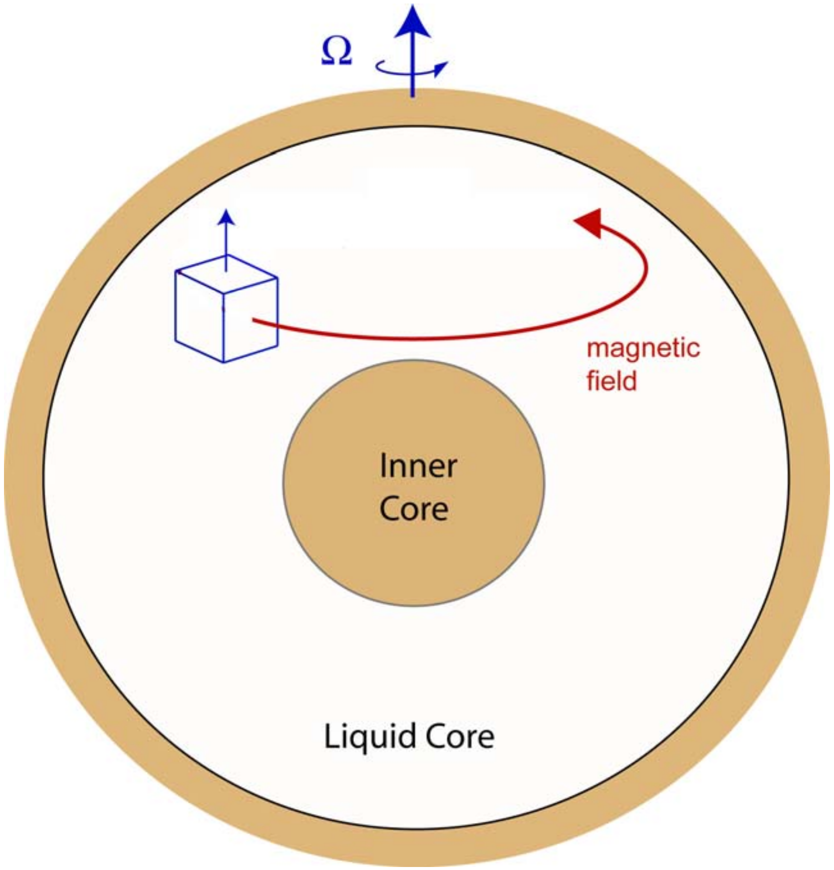


Glatzmaier & Roberts (1996)



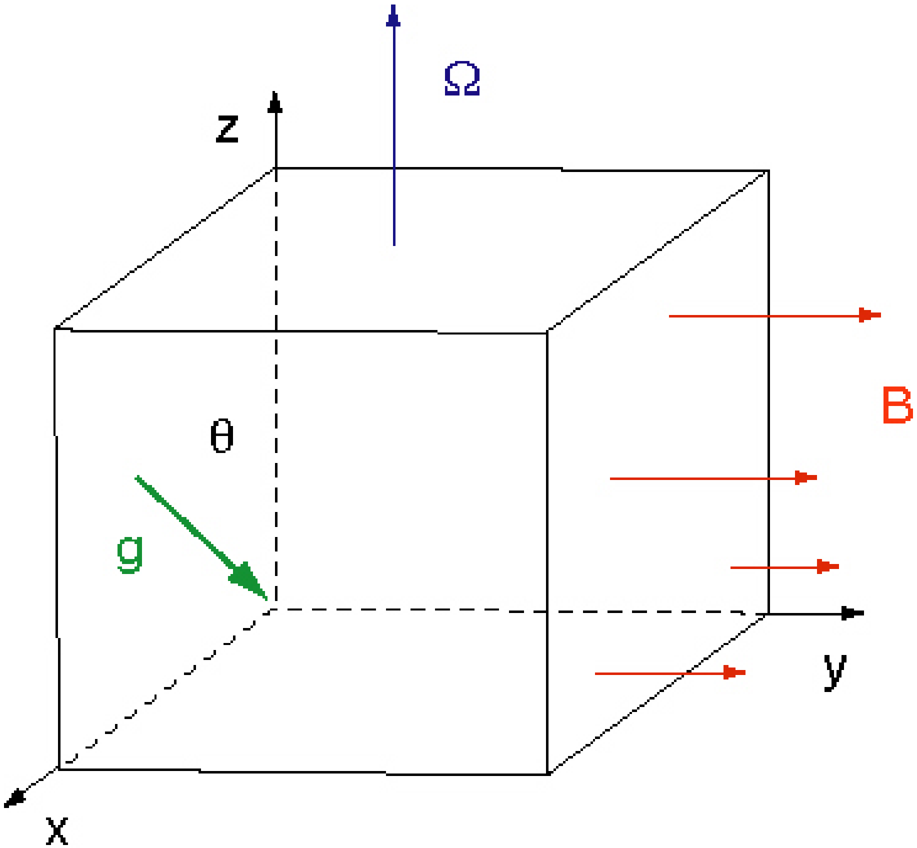
Kuang & Bloxham (1997)

Structure of Small-Scale Flow



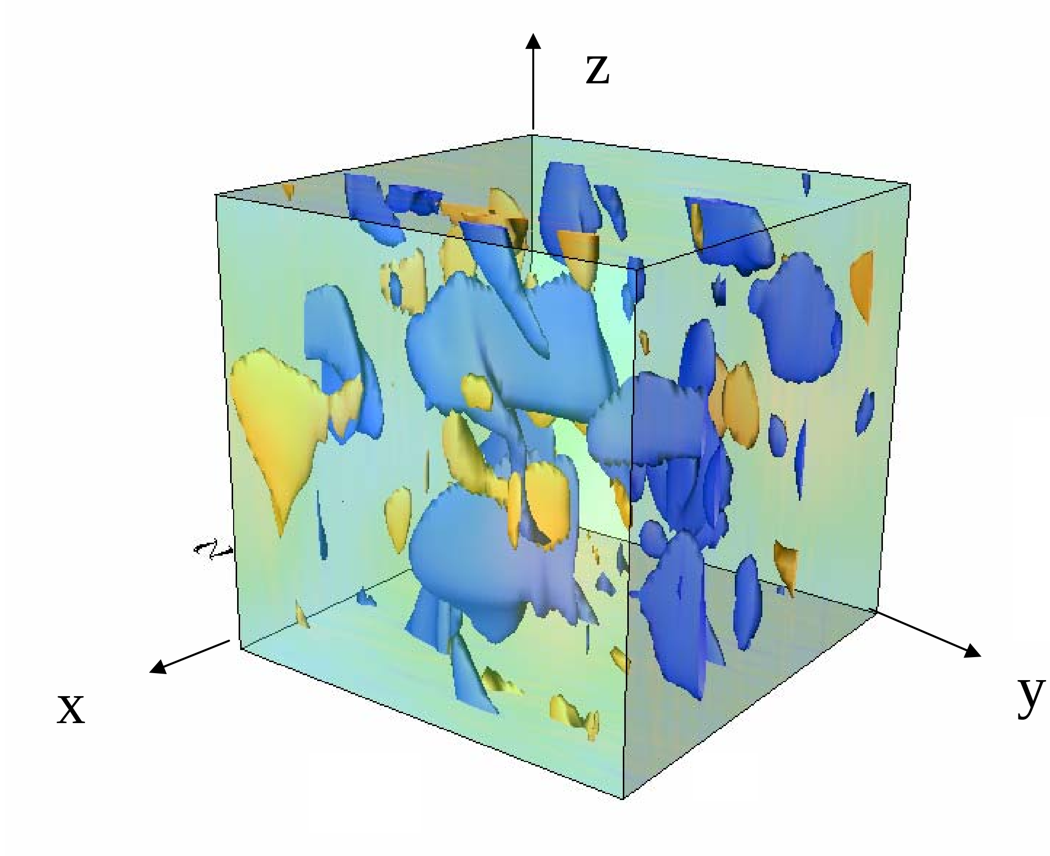
use realistic properties in a small (10 km)^3 volume

Model



Grid 256 x 128 x 64

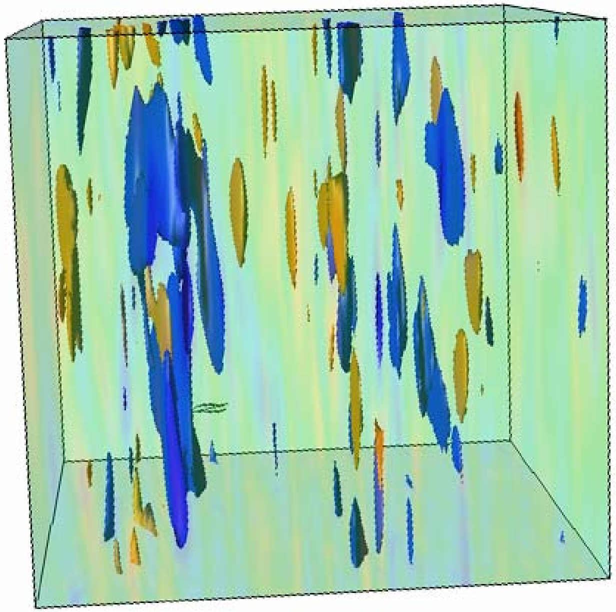
Temperature Field



Grid 256 x 128 x 64

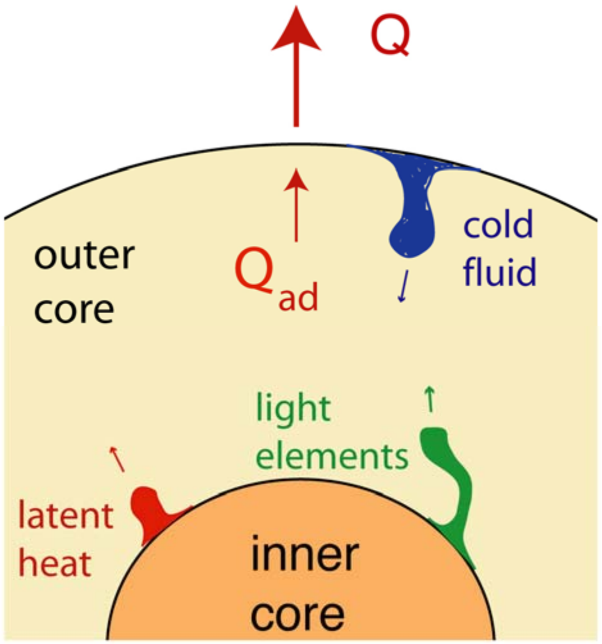
Temperature Field

View along magnetic field lines

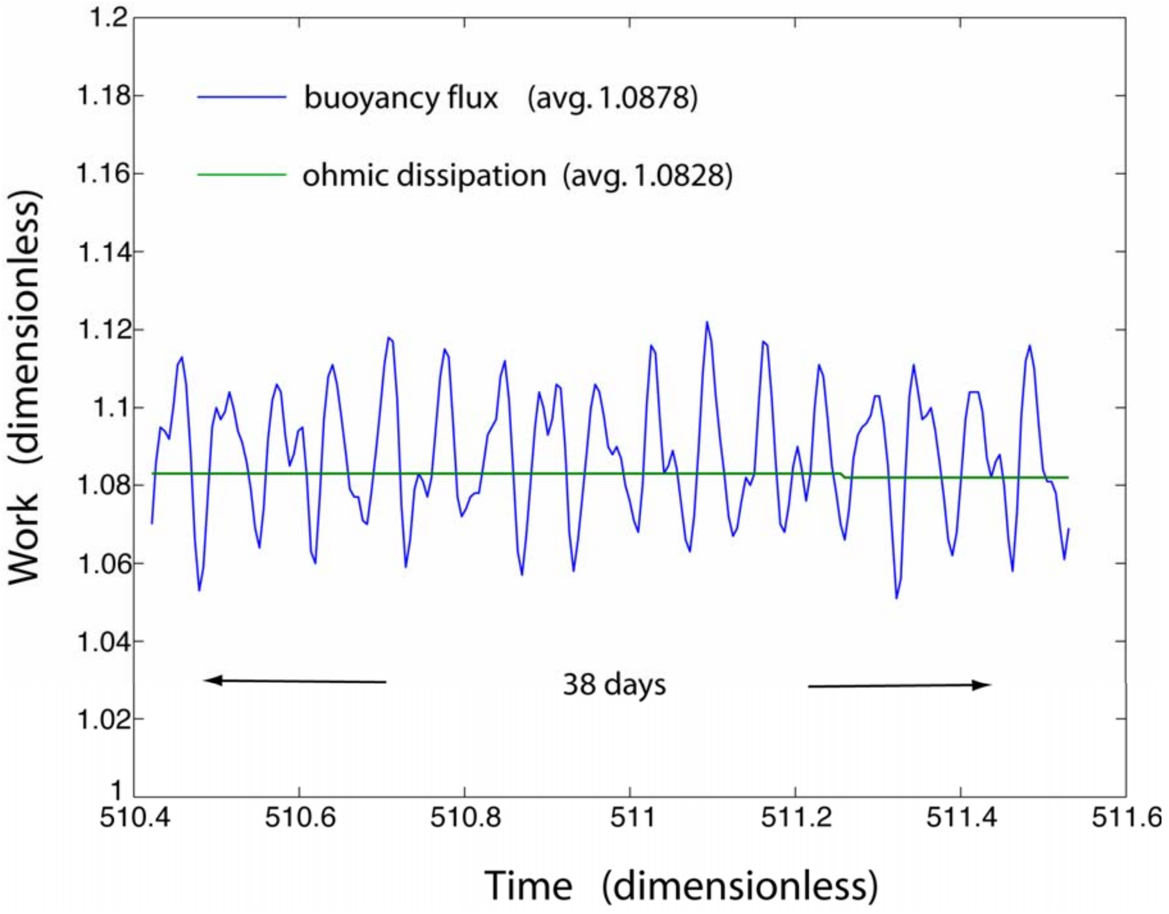


A problem with physical properties in models

Physical Processes



Energy Balance



Eddy Diffusivity Model

Approximate SGS heat flux using

$$I = \widetilde{VT} - \widetilde{V}\widetilde{T} = -K\nabla\widetilde{T}$$

where $K \approx vl$

Result

$$\frac{\partial \widetilde{T}}{\partial t} = -\nabla \cdot (\widetilde{V}\widetilde{T}) + (\kappa + K)\nabla^2 \widetilde{T}$$

Description of Problem

1. Momentum equation (ma = f)

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + 2\Omega \times \mathbf{V} = -\nabla P + \frac{\Delta \rho}{\rho}\mathbf{g} + \frac{1}{\mu\rho}\mathbf{B} \cdot \nabla \mathbf{B} + \nu \nabla^2 \mathbf{V}$$

Coriolis buoyancy magnetic viscous

Newton

Character of Flow

$$\frac{\text{viscous}}{\text{Coriolis}} \approx \frac{\nu}{2\Omega L^2} \approx 10^{-16}$$

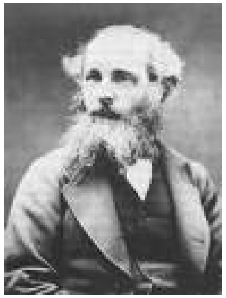
$$\frac{\text{inertia}}{\text{Coriolis}} \approx \frac{V}{2\Omega L} \approx 10^{-7}$$

Description of Problem

2. Induction Equation

$$\frac{\partial B}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

$$\frac{\text{induction}}{\text{diffusion}} \approx \frac{VL}{\eta} \approx 10^3$$



Maxwell

3. Heat Equation

$$\frac{\partial T}{\partial t} = -\nabla \cdot (\mathbf{V}T) + \kappa \nabla^2 T$$

Fourier