Static vacuum black hole solutions in Einstein Gauss Bonnet gravity

(hep-th/0508118 and gr-qc/0409005 0503117,0510069, joint with Reinaldo Gleiser)

Outline

- Lovelock theory and Einstein Gauss Bonnet gravity.
- Static black holes in D>4 GR, horizons of constant curvature and more general Einstein manifolds (EM) as possible horizons.
- Limitations to the horizon geometry in EGB.
- Stability of EGB BHs with constant curvature horizons.

Lovelock theory, EGB gravity

Lovelock gravity is given by

$$G_b{}^a = \sum_{p=0}^{[(d-1)/2]} c_p \, G_{(p)b}{}^a$$

$$G_{(p)b}{}^a = g_{[b}^{[a} R^{i_1 i_2}{}_{i_1 i_2} R^{i_3 i_4}{}_{i_3 i_4} \ldots\ldots R^{i_{2p-1} i_{2p}]}{}_{i_{2p-1} i_{2p}]}$$

G_{ab} is the most general symmetric, divergency free tensor that can be made using up to second derivatives of the metric

$G_{(0)ab} = g_{ab}$ and $G_{(1)ab} = R_{ab} - \frac{R}{2} g_{ab}$

Einstein Gauss Bonnet gravity is the special case $c_p = 0, p > 2$

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Gauss Bonnet corrections to GR

The **Gauss-Bonnet** tensor is:

$$G_{(2)b}{}^a = R_{cb}{}^{de} R_{de}{}^{ca} - 2 R_d{}^c R_{cb}{}^{da} - 2 R_b{}^c R_c{}^a + R R_b{}^a - \frac{1}{4} \delta_b^a \left(R_{cd}{}^{ef} R_{ef}{}^{cd} - 4 R_c{}^d R_d{}^c + R^2 \right)$$

It shows up as a low energy string theory correction to Einstein's gravity

$$\alpha \, G_{(2)b}{}^a + R_b{}^a - \frac{1}{2} R g_b{}^a + \Lambda g_b{}^a = T_b{}^a$$

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Static black holes in D>4 GR

The ansatz $ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2\bar{g}_{ij}dx^i dx^j$

($\bar{g}_{ij}$ the metric of Σ_n, an n-dimensional *"horizon manifold"*)

inserted in D=n+2 (vacuum) Einstein's equations (with c.c. Λ), gives

$R_{ij} = \kappa(n-1)g_{ij}$ $\qquad$ $g(r) = 1/f(r)$

and $f(r) = \kappa - \frac{2\mu}{nr^{n-1}} - \frac{2\Lambda r^2}{n(n+1)}$

There is a large zoo of static black hole solutions in higher dimensional Einstein gravity

About Einstein manifolds

- **n=3 (D=5):** the EM condition implies Σ_3 is a *constant curvature manifold:*

$$\bar{R}_{ij}{}^{kl} = \kappa\left(\bar{g}_i{}^k\,\bar{g}_j{}^l - \bar{g}_i{}^l\,\bar{g}_j{}^k\right)$$

- **n>3 (D>5):** the EM condition (EM) implies that

$$\bar{R}_{ij}{}^{kl} = \bar{C}_{ij}{}^{kl} + \kappa\left(\bar{g}_i{}^k\,\bar{g}_j{}^l - \bar{g}_i{}^l\,\bar{g}_j{}^k\right),$$

Static vacuum BHs in EGB gravity

Constant curvature horizon BHs in EGB
(JTWheeler '85,...)

If the horizon Σ_n is a constant curvature manifold there are solutions:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\bar{g}_{ij}dx^i dx^j$$

$$f(r) = \kappa + \frac{r^2}{\alpha(n-1)(n-2)}\left(1 \pm \sqrt{1 + \frac{4\alpha(n-1)(n-2)}{n(n+1)}\left[\frac{\mu(n+1)}{r^{n+1}} + \Lambda\right]}\right)$$

where μ is an integration constant

Note that $\mu=0$ gives 2 cosmological solutions, the constants of the theory can be fine tuned to give a single effective cosmological constant (degenerate theory)

In the $\alpha \longrightarrow 0$ limit, the (-) branch reduces to the GR solution

$$f(r) = \kappa - \frac{2\mu}{nr^{n-1}} - \frac{2\Lambda r^2}{n(n+1)} + \mathcal{O}(\alpha)$$

EM horizon BHs in EGB
(Dotti-Gleiser, PLB'05)

If the horizon Σ_n is an EM of non constant curvature, its Weyl tensor must satisfy the ***Horizon Condition***:

$$\sum_{kls} \bar{C}_{ki}{}^{ls}\bar{C}_{ls}{}^{kj} = \theta\delta_i{}^j$$

where θ **is a constant.**

$$f(r) = \kappa + \frac{r^2}{\alpha(n-1)(n-2)}\left(1 \pm \sqrt{1 + \frac{4\alpha\,(n-1)\,(n-2)}{n(n+1)}\left[\frac{\mu(n+1)}{r^{n+1}} - \frac{\alpha(n+1)\theta}{4(n-3)r^4} + \Lambda\right]}\right)$$

If the EGB theory is non degenerate, the horizon **has to be Einstein** (Troncoso,Oliva & Dotti, work in progress)

Obstructions on the horizon geometry in EGB gravity

- The horizon condition (HC)

  $$\sum_{klm} \bar{C}_{ki}{}^{lm}\bar{C}_{lm}{}^{kj} = \theta\delta_i{}^j$$

  is both an algebraic and a differential constraint on the Weyl tensor since $\nabla_k\theta = 0$
- The algebraic constraint is *always* met if n=4 (D=6), although in general θ is non constant.
- For n>4 (D>6), the algebraic constraint does not hold in general.

Example: Bohm metrics

The Bohm metrics EM locally given by

$$ds^2 = d\rho^2 + a(\rho)^2 d\Omega_p^2 + b(\rho)^2 d\Omega_q^2$$

$$0 \leq \rho \leq \rho_f \quad a(\rho), b(\rho) > 0$$

They can be extended to S^{p+q+1} or $S^{p+1} \times S^q$

If appropriate boundary conditions are imposed to a(ρ) and b(ρ)

There is a countably infinite family of non trivial metrics on S^{p+q+1} and $S^{p+1} \times S^q$ besides the standard ones.

Any Bohm horizon is possible in GR, although the resulting BHs are unstable under linear perturbations of the metric (Gibbons, Hartnoll & Pope, PRD '03).

None of the (non-trivial) Bohm metrics satisfies the HC! There are no BHs in EGB with Bohm horizons.

The only BH with a non constant curvature horizon found (so far) has S^p x S^p as a horizon (then even spacetime dimension)

First conclusion: GB term severely restricts the possibilities for the horizon geometry of a static BH

Do we get further GB restrictions from linear stability considerations ?

Linear stability of EGB BHs with CC horizons

(with R.Gleiser, gr-qc/0510069, gr-qc/0503117, gr-qc/0409005 - CQG & PRD 2005)

Linear perturbations $g_{ab} \to g_{ab} + h_{ab}$ around

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 \bar{g}_{ij} dx^i dx^j$$

can be decomposed following Kodama et al:

$$h_{ab} = c_1 h_{ab}^{(tensor)} + c_2 h_{ab}^{(vector)} + c_3 h_{ab}^{(scalar)}$$

Example: Tensor mode

$$h_{tt}^{(tensor)} = h_{tr}^{(tensor)} = h_{rr}^{(tensor)} = 0$$

$$h_{ij}^{(tensor)}(t,r,x) = r^2 \phi(r,t) \bar{h}_{ij}(x)$$

where the Σ_n tensor $\bar{h}_{ij}$ satisfies

$$(\bar{\triangle} + k_T^2)\bar{h}_{ij} = 0, \quad \bar{\nabla}^i \bar{h}_{ij} = 0, \quad \bar{g}^{ij}\bar{h}_{ij} = 0$$

This mode has as one unknown function: $\phi(r,t)$

Equivalent Schrödinger equation

For *every mode*, the linearized equations reduce to a single equation on one unknown function ϕ:

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial \Phi}{\partial r^{*2}} + V(r^*)\Phi = 0 \qquad (dr^*/dr = 1/f)$$

Separating variables $\Phi(r^*,t) = \phi(r^*)\, e^{\omega t}$ we get a stationary Schrödinger equation *(the potentials are different for different modes and depend on the harmonic number)*

$$\mathcal{H}\phi \equiv -\frac{\partial^2 \phi}{\partial r^{*2}} + V\phi = -\omega^2 \phi \equiv E\phi$$

Instability iff E < 0

The results: tensor mode

$$V(r) = q + \frac{f}{K}\frac{d}{dr}\left(f\frac{dK}{dr}\right)$$

$$K(r) = r^{n/2-1}\sqrt{r^2 + \alpha(n-2)\left((n-3)(\kappa - f) - r\frac{df}{dr}\right)}$$

$$q = \left(\frac{f(2\kappa + k_S^2)}{r^2}\right)\left(\frac{(1-\alpha f'')r^2 + \alpha(n-3)\left[(n-4)(\kappa - f) - 2rf'\right]}{r^2 + \alpha(n-2)\left[(n-3)(\kappa - f) - rf'\right]}\right)$$

S-deformation analysis

We are interested in a lower bound for the spectrum of

$$A := -\frac{d^2}{dr^{*2}} + V$$

The expectation value of this operator for a test function ϕ can be written:

$$(\phi, A\phi) = \int_{r_1^*}^{r_2^*}\left(|D\phi|^2 + \tilde{V}|\phi|^2\right)dr^*$$

$$D = \frac{d}{dr^*} + S, \qquad \tilde{V} = V + f\frac{dS}{dr} - S^2$$

Choosing $S = -f \frac{d}{dr} \ln(K)$ one gets

$$(\phi, A\phi) = \int_{r_1^*}^{r_2^*} |D\phi|^2 dr^* + \int_{r_1}^{r_2} \frac{|\phi|^2 q}{f}\, dr$$
$$= \int_{r_1^*}^{r_2^*} |D\phi|^2 dr^* + (2\kappa + k_S^2) \int_{r_1}^{r_2} \frac{|\phi|^2 H}{r^2}\, dr$$

Factorization of S-deformed potential tells us that a black hole is static under tensor perturbations if and only if $H(r) \geq 0$ for all r

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Any interesting tensor instability ?

- Spherical, asympotically euclidean BHs are stable in GR under tensor perturbations if $D \neq 6$
- A tensor mode instability was found for *low mass* 6D spherical, AE, EGB BHs

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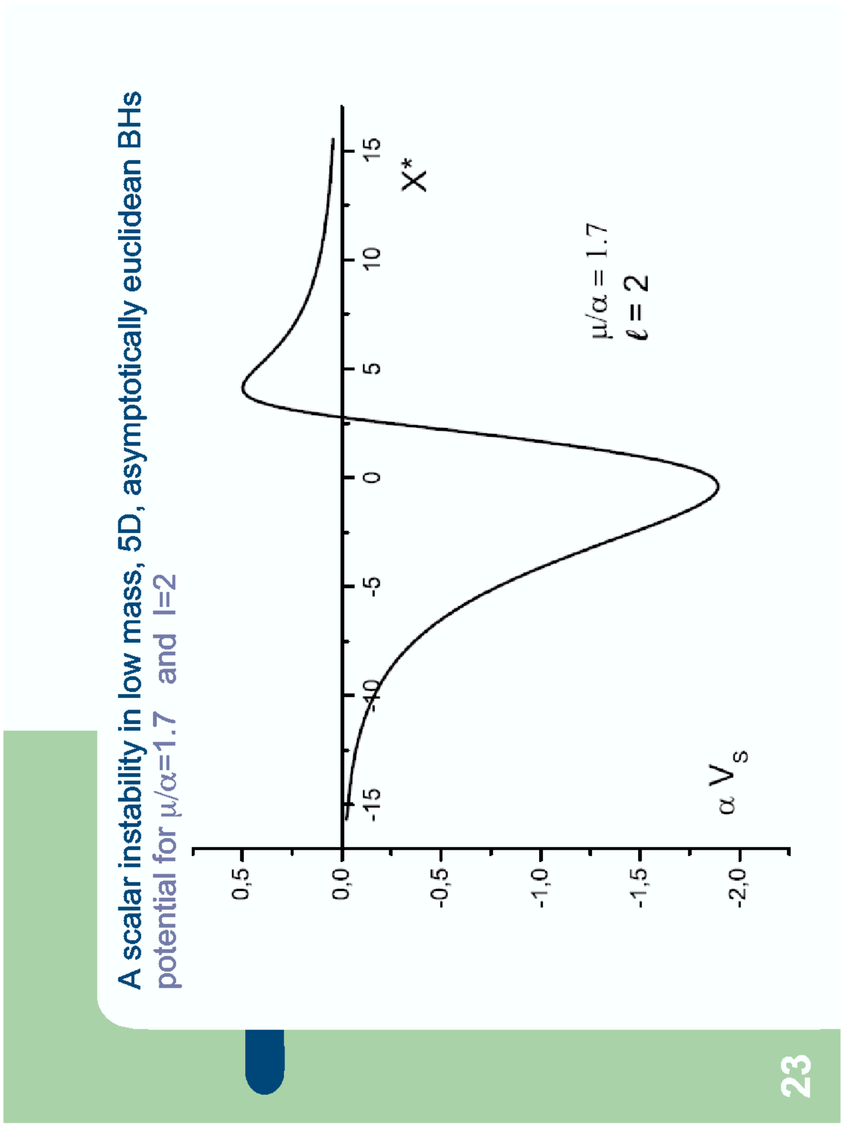
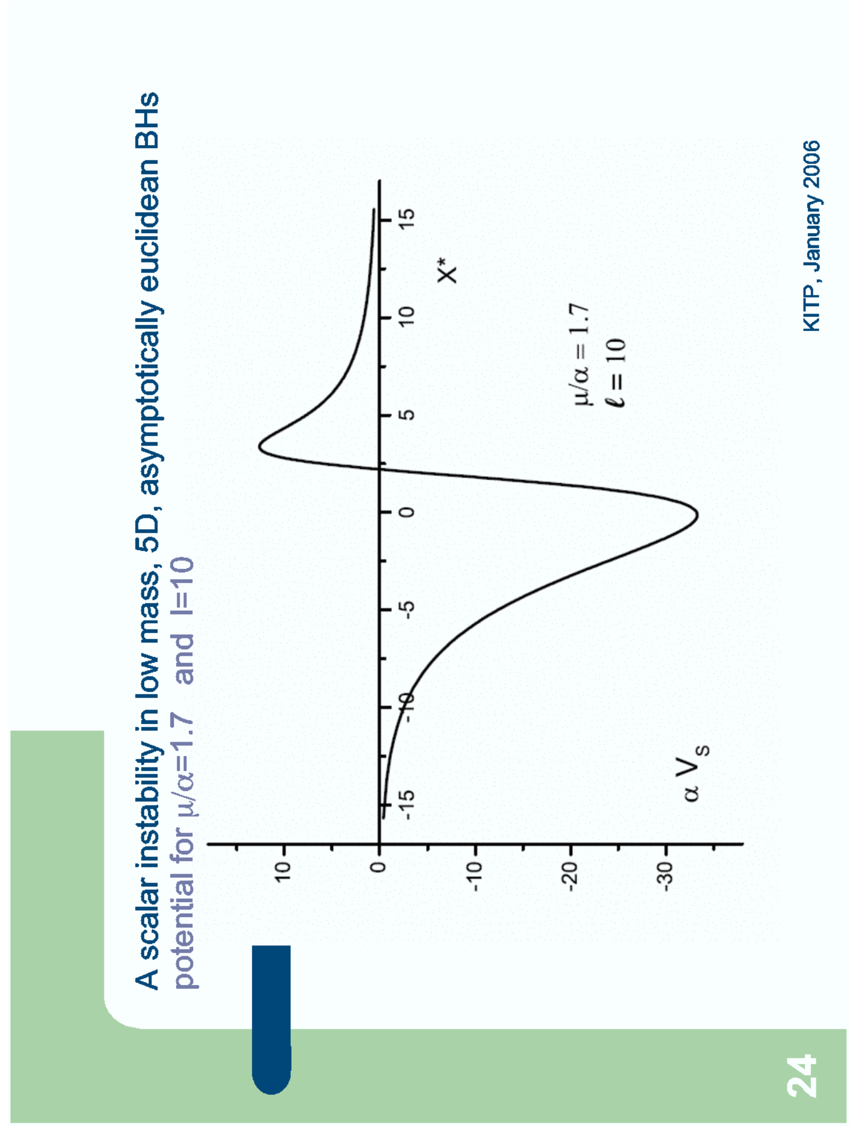
The results: vector mode

- Equations much more complicated than tensor mode, had to interpolate in dimension.
- Potential can be S-deformed into one which factors out the harmonic eigenvalue.
- No interesting BH instability found (as happens in GR)

The results: scalar mode

- Equations much more complicated than tensor and vector modes, had to interpolate.
- Cannot write compact form for the potential.
- Harmonic eigenvalue appears in rather involved non linear way
- Cannot S-deform into something reasonable.
- Can only draw curves....
- Found an instability in D=5, for low mass


A scalar instability in low mass, 5D, asymptotically euclidean BHs
potential for μ/α=1.7 and l=2
μ/α = 1.7
ℓ = 2
X*
α V_S
23


A scalar instability in low mass, 5D, asymptotically euclidean BHs
potential for μ/α=1.7 and l=10
μ/α = 1.7
ℓ = 10
X*
α V_S
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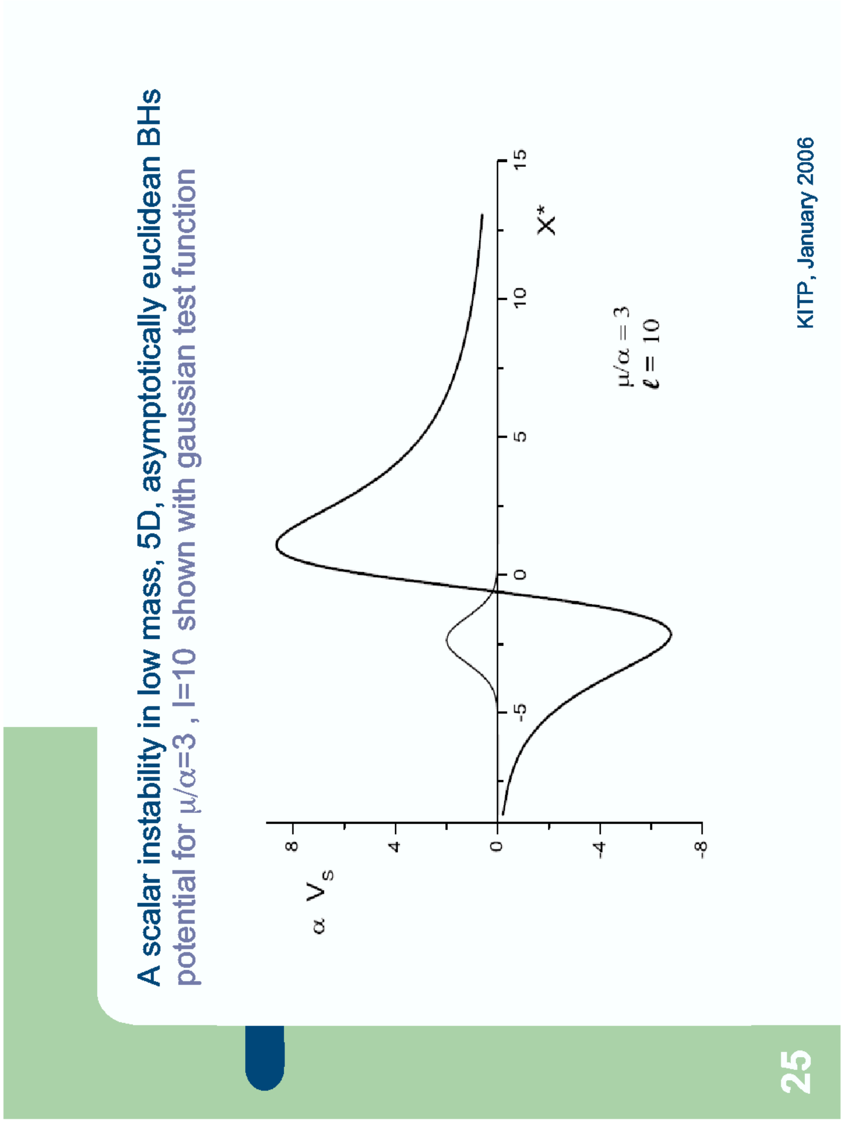
A scalar instability in low mass, 5D, asymptotically euclidean BHs
potential for μ/α=3 , l=10 shown with gaussian test function
α V_S
X*
μ/α = 3
ℓ = 10
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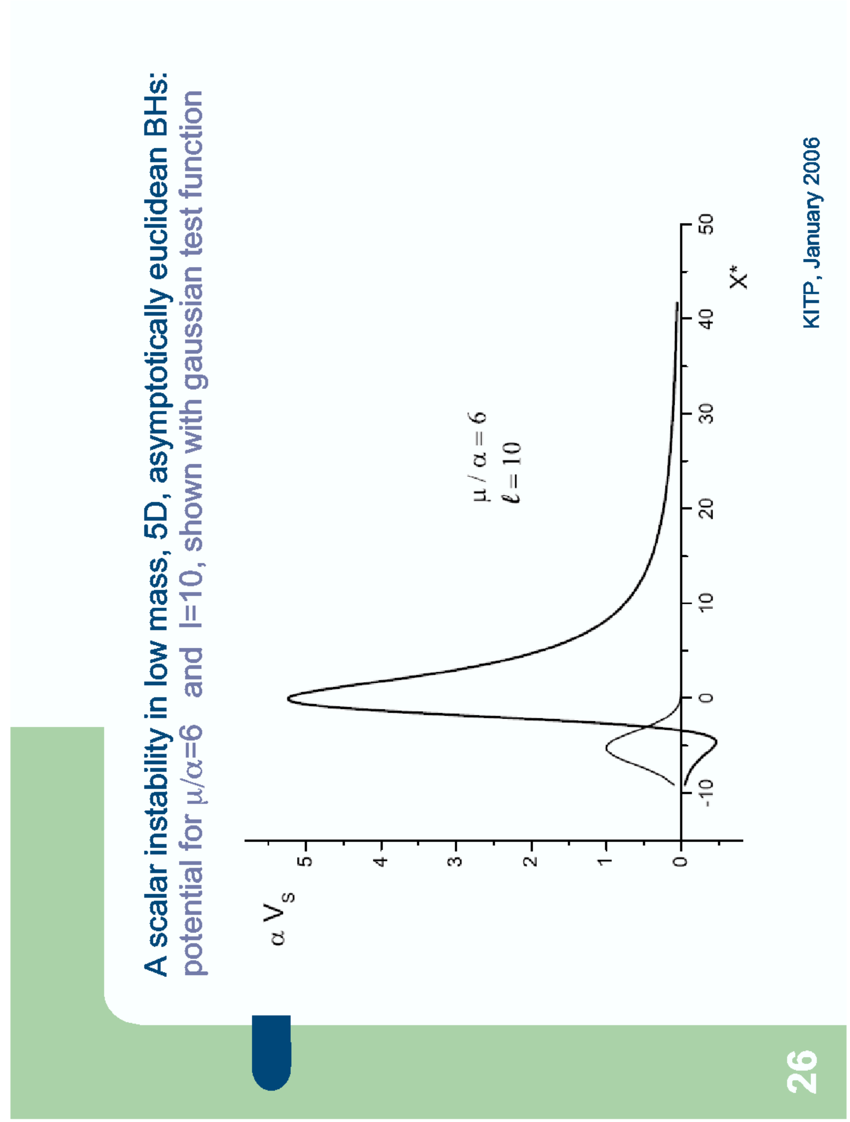
A scalar instability in low mass, 5D, asymptotically euclidean BHs:
potential for μ/α=6 and l=10, shown with gaussian test function
α V_S
X*
μ / α = 6
ℓ = 10
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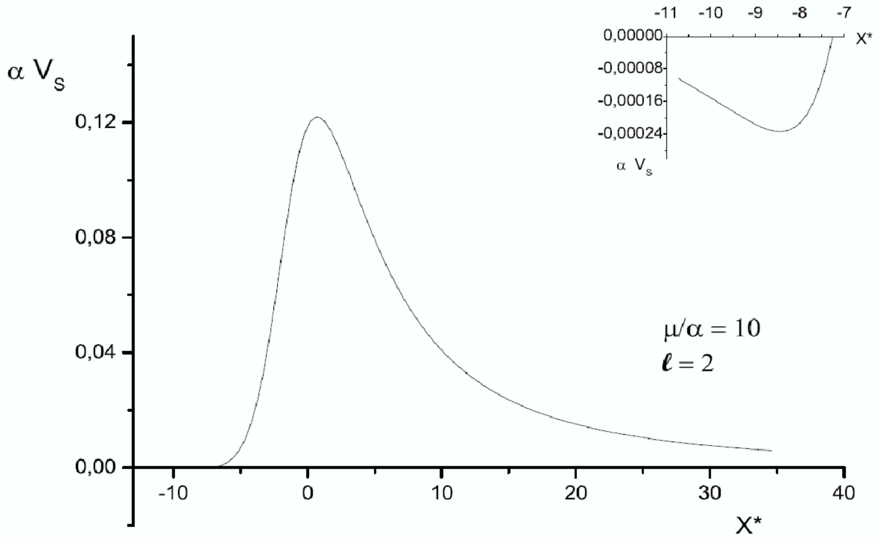
A scalar instability in low mass, 5D, asymptotically euclidean BHs: potential for μ/α=10 and l=2 shows stability for high enough mass



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Conclusions

- Given D>4 GR BH solutions, it is natural to look for analogous solutions in EGB. After all, D>4 is motivated by string theory, and EGB contains the dominant string correction to GR
- Found that most BHs with Einstein manifold horizons *are not allowed* in EGB
- Found that simplest BHs (spherical, AE) are *not stable* for low mass in D=5 (scalar instability) and D=6 (tensor instability)

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Vector mode

$$h_{ab}^{(vec)} = 0 \; , \; h_{ai}^{(vec)} = r\, f_a V_i \; , \; h_{ij}^{(vec)} = 2r^2 H_T V_{ij}$$

where the Σ_n vector V_i satisfies $(\bar{\triangle} + k_V^2) V_i = 0 \quad \bar{\nabla}_i V^i = 0$

and $$V_{ij} \equiv -\frac{1}{2k_V}\left(\bar{\nabla}_i V_j + \bar{\nabla}_j V_i\right)$$

Contains three unknown functions:

$$f_r(r,t) \quad f_t(r,t) \quad H_T(r,t)$$

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Scalar mode

$$h_{ab}^{(scalar)} = \mathcal{F}_{ab} S, \; h_{ai}^{(scalar)} = r \mathcal{F}_a S_i$$
$$h_{ij}^{(scalar)} = 2r^2 (H_L \bar{g}_{ij} S + H_T S_{ij})$$

where the Σ_n scalar S satisfies $(\bar{\triangle} + k_S^2) S = 0$

and $S_i \equiv -\frac{1}{k_S} \bar{\nabla}_i S, \quad S_{ij} \equiv \frac{1}{k_S^2} \bar{\nabla}_i \bar{\nabla}_j S + \frac{1}{n} \bar{g}_{ij} S$

This mode has seven unknown functions:

$$\mathcal{F}_{ab}, \mathcal{F}_a, H_L, H_T$$

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