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Algebraic topology and differential forms I

Remark: Topological constructions answering questions in physics usually give an answer that is too crude:

Example: If Σ - 2-manifold
S - parameter space

and

$$\alpha \in H^3(\Sigma \times S)$$

- family of cohomology classes on Σ parameterized by S, then α most naturally gives rise to

$$\int_\Sigma \alpha \in H^1(S, \mathbb{Z})$$

i.e. $\int_\Sigma \alpha$ - homotopy class of maps

$$S \to S^1$$

However in physics this is not enough. We would like to have an actual map rather than a homotopy class.

The goal of the lectures is to describe a refinement of topology that will produce a more 'physical' answer.

Based on a joint work with I. Singer, D. Freed

math. DG/... : Quadratic functions.

Differential functions:

If M - smooth manifold

X - topo space

$\iota \in Z^n(X, \mathbb{R})$ - n-cocycle on X

then we define a differential function from M to (X, ι) as a triple

$$(c, h, \omega) : M \longrightarrow (X, \iota)$$

where

$c : M \to X$ - continuous map

$h \in C^{n-1}(M, \mathbb{R})$

$\omega \in \Omega^n(M)$

$$\boxed{\delta h = \omega - c^*(\iota)}$$

We would like to do algebraic topology starting with differential functions rather than functions

For this we need to understand how differential functions vary. It is convenient to introduce some simplicial or combinatorial machiner as a tool.

Recall: If Y is a space, then the topological invariants of Y are contained into a sequence of sets + maps between them:

$$\mathrm{Sing}_0(Y) = \{\text{pts of } Y\}$$

$$\mathrm{Sing}_1(Y) = \{\text{maps } \Delta^1 \to Y\}$$

$$\mathrm{Sing}_2(Y) = \{\text{maps } \Delta^2 \to Y\}$$

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and the maps between $\mathrm{Sing}_i(Y)$ and $\mathrm{Sing}_k(Y)$ tell us how different simplices are glued to each other.

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The collection of sets $Sing_i$ together with the maps $Sing_i \to Sing_k$ (faces + degeneracies) is what is known as a simplicial set.

Simplicial sets are useful devices and we will use them to make the differential functions into a space

Def: The space of differential functions

$$(X, \iota)^M$$

is a space with n-simplices

$$(c, h, \omega) : M \times \Delta^n \longrightarrow (X, \iota)$$

Furthermore this space is filtered and

$$\text{filt}_k (X, \iota)^M = \text{Space whose n-simplices are}$$

$$(c, h, \omega) : M \times \Delta^n \to (X, \iota)$$

with

$$\omega \in \bigoplus_{j \leq k} \Omega^i(M) \otimes \Omega^j(M)$$

Note: We need the filtration as part of the structure. If we forget about the filtration, then the spaces of differential functions will not give anything more than the spaces of ordinary functions.

We would like to understand the space of differential functions and its topological invariants in examples.

Recall: One of the most basic topo invariants of a space is its fundamental group or more generally its fundamental groupoid

(If Y - space then the fundamental groupoid $\pi_{\leq 1}(Y)$ of is a category with

objects = points of Y

maps = paths in Y connecting points

Composition law = composition of paths

Examples:

- $X = \mathbb{C}P^\infty$

  $\iota \in Z^2(\mathbb{C}P^\infty, \mathbb{R})$ - representing c_1 of the universal bundle

Then for any manifold M we have

$$\pi_{\leq 1}\, \mathrm{filt}_2\,((X, \iota)^M)$$

is the category with

<u>Objects</u>: $U(1)$ bundle on M with connection

(if this object is given by

$$(c, h, \omega) : M \to (X, \iota)$$

then

$$\omega \in \Omega^2(M)$$

is the curvature 2-form of our connection) maps: principal bundle maps modulo homotopy

This is equivalent to

$\pi_{\leq 1}$ Maps $(M, \mathbb{C}P^\infty)$

$\uparrow$

usual purely topological notion of maps

In other words if we take the second stage in the filtration of the space of differential functions, then we recover ordinary topology.

But the data of the filtration carries more info.

If we look at $\pi_{\leq 1}$ $\mathrm{filt}_1((X, i)^M)$ we get a groupoid with

objects: $U(1)$ bundles with connection

maps: Principal bundle maps with no equivalence on them

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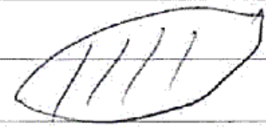
Remark: To see that this are the maps we look at two homotopic paths



and a disc Δ bounding them

Then the condition that we are in $filt_1$ says that

ω has at most a 1-form component on Δ

Finally we can look at

$$\pi_{\leq 1}\, filt_0\left(\left(X, i\right)^M\right)$$

which is now a groupoid with

objects: principal $U(1)$ bundles with connections

maps: horizontal (connection preserving) maps

Note: To see that these are the right maps note that the condition for being in $\check{H}_0$ says that the curvature form in

$$\omega \in \Omega^2(M \times [0,1])$$

must be constant along $[0,1]$

This implies that the holonomies of our connection do not change along $[0,1]$.

Generalization: Take $X = K(\mathbb{Z}, n)$ and let $\iota \in Z^n(X, \mathbb{R})$ be a representative of the generator of $H^n(X, \mathbb{Z})$.

Now

$$\pi_0(X^M) = [M, X] = H^n(M, \mathbb{Z})$$

$$\pi_0(\check{H}_0(X, \iota)^M) = \text{Cheeger-Simons cohomology group } \check{H}^{n-1}(M)$$

$$= \text{Deligne cohomology group } H^{n,n}(M)$$

This group sits in a short exact sequence

$$0 \to H^{n-1}(M, \mathbb{R}/\mathbb{Z}) \longrightarrow H^{n,n}(M) \to \Omega^n_{cl,\mathbb{Z}}(M) \to 0$$

closed forms with integral periods.

Again we see that filt_k carries more refined geometric information than the ordinary topological mapping spaces.

In fact we can take X = space representing some kind of cohomology theory E.

For instance: • $X = K(\mathbb{Z}, n)$ represents ordinary cohomology H^n.

• X = space of Fredholm operators = classifying space for K-theory

Now looking at π_i of $(X, i)^M$ we get a differential version of

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the cohomology theory E.
(We will denote this theory by
$\check{E}(n)(M)$)

Example: Differential K-theory = the K-theory of vector bundles with super connections

Digression: Descriptions of $\check{H}^{n-1}(M) = H^{n,n}(M)$.

- The original Cheeger-Simons definition is:

$\check{H}^{n-1}(M)$ = pairs (χ, ω) consisting of a homomorphism $Z_{n-1}(M) \xrightarrow{\chi} \mathbb{R}/\mathbb{Z}$

$(\supset) \; \omega \in \Omega^n(M)$

s.t.

$$\chi(\partial N) = \int_N \omega$$

- Deligne's definition

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Consider

$$\mathbb{Z}(n) := \mathbb{Z} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^{n-1}$$

– de Rham complex stupidly truncated at level <u>n-1</u>.

Then

$$H^{m,n}(M) := H^m(M, \mathbb{Z}(n))$$

for all m.

- <u>Definition coming from the differential functions point of view</u>

Consider a complex $C(n)^{\bullet}(M)$ given by

$$C(n)^k(M) := \text{set of } (c, h, \omega)$$

$$c \in C^k(M, \mathbb{Z})$$

$$\omega \in \Omega^k(M) \qquad \omega = 0, \ k < n$$

$$h \in C^{k-1}(M, \mathbb{R})$$

with a differential

$$\delta(c, h, \omega) = (\delta c, \delta h - \omega + c, d\omega)$$

$$\left(\delta(c, h, \omega) = 0 \iff \begin{array}{l} c, \omega - \text{closed} \\ \delta h = \omega - c \end{array} \right)$$

Now $\begin{pmatrix} m\text{-th cohomology} \\ \text{group of } C(n)^*(M) \end{pmatrix} = H^{m,n}(M)$,

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The 'ordinary' differential cohomology $H^{m,n}(M)$ as we saw correspond to the differential function spaces from M to Eilenberg - MacLane spaces, For these differential cohomolo we have all the usual apparatus of cohomology (cup product, integration, ...).

Let's illustrate this on an example:

Let G - compact Lie group

$$X = BG$$

$$\iota \in Z^4(BG, \mathbb{R})$$

Now given a principal G-bundle with connection on M gives rise (Chern-Weil) to a differential function $M \to (BG, \iota)$

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Caution: If $G \neq U(1)$ then a differential function $M \to (BG, \iota)$ contains less information than a connection.

Fix a principal G-bundle P on M then we get a map

$$\mathcal{A}_P \longrightarrow (BG, \iota)^M$$

$\uparrow$
space of connections

In fact this map lands in $fil t_0 (BG, \iota)^M$

Let

$$(c, h, \omega) : M \times \mathcal{A}_P \to (BG, \iota)$$

be the corresponding differential function

If $\dim M = 2$, $G = U(1) \Rightarrow$

$$\int_M (c^*\iota, h, \omega) = \text{differential function } \mathcal{A}_P \to (\mathbb{CP}^\infty, \iota)$$
$$= U(1) \text{ bundle with connection on } \mathcal{A}_P$$

This leads to a cocycle refinement of the classical Chern-Simmons theory as worked out by Gawedski.