Mike Hopkins 07-15-03

Lecture 2

Last time we used simplicial sets and their relation to topological spaces to endow the set of differential functions $(X, \bar{\iota})^M$ with the structure of a topological space.

Let's consider some other examples of the same method:

- Suppose that we would like to make a space out of the set of all closed n-forms on a manifold M. Following the recipe from last lecture we can take the simplicial set whose K-simplices are

$$\Omega^n_{cl}(M \times \Delta^k)$$

Where Δ^k - the standard k-simplex.

Now the simplicial set

$$\cdots \Omega^n(M \times \Delta^2) \; \substack{\rightarrow\\\rightarrow\\\rightarrow} \; \Omega^n_{cl}(M \times [0,1]) \; \substack{\rightarrow\\\leftarrow\\\rightarrow} \; \Omega^n_{cl}(M) \qquad (*)$$

can on one hand be viewed as a topological space (e.g. via a geometric realization) or alternatively can be viewed as a complex of abelian groups. Indeed: applying $\int_{\Delta^k}$ to the simplicial abelian group $(*)$ we get a chain complex of abelian groups which is the truncated de Rham complex

$$\Omega^0(M) \longrightarrow \Omega^{n-2}(M) \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n_{cl}(M)$$

- Similarly: the 'space' of n-cocycles on X with values in A has an abelian model

$$C^0(X, A) \to \cdots \to C^n_{cl}(X, A)$$

which is the n-truncated chain complex computing the cohomology of X.

Note: Here we used the standard fact (Dold-Puppe) that the categories of simplicial abelian groups and of chain complexes are equivalent

3.

Following the pattern of the previous examples we would like to have a chain complex interpretation of the space of differential functions

$$M \longrightarrow (K(\mathbb{Z}, k), \iota)$$

Ordinary cohomology is given by complex $C^\bullet(0)(M$ defined as the fiber product

$$\begin{array}{ccc} C(0)^\bullet(M) & \longrightarrow & \Omega^\bullet \\ \downarrow & & \downarrow \\ C^\bullet(M, \mathbb{Z}) & \longrightarrow & C^\bullet(M, \mathbb{R}) \end{array}$$

Explicitly

$$C(0)^k(M) = \underset{c}{C^k(M, \mathbb{Z})} \times \underset{h}{C^{k-1}(M, \mathbb{R})} \times \underset{\omega}{\ }$$

and

$$\delta(c, h, \omega) = (\delta c, \omega - c - \delta h, d\omega)$$

$C^\bullet(0)(M)$ =: complex of differential cochains

$H(0)^\bullet(M) :=$ cohomology of $C(0)^k(M)$
$= H^\bullet(M, \mathbb{Z})$

The data of the space

$$(K(\mathbb{Z}, k), \iota)^M$$

is captured by the complex $C^\bullet(0)(M)$ by subcomplexes $C^\bullet(k)(M)$.

Here $C^\bullet(n)(M)$ is defined as the fiber product

$$\begin{array}{ccc} C(n)^\bullet(M) & \longrightarrow & \Omega^\bullet_{\geq n} \\ \downarrow & & \downarrow \\ C^\bullet(M, \mathbb{Z}) & \longrightarrow & C^\bullet(M, \mathbb{R}) \end{array}$$

We set

$H(n)^\bullet(M)$ = cohomology of $C(n)^\bullet(M)$

Note that

$$H(n)^k(M) = \begin{cases} H^k(M,\mathbb{Z}) & k>n \\ H^{k-1}(M,\mathbb{R}/\mathbb{Z}) & k<n \end{cases}$$

and that

$$H^{n-1}(M)\otimes\mathbb{R}/\mathbb{Z} \to H(n)^n(M) \to A^n(M)$$

Here $A^n(M) = \{(x,\omega) \in H^n(M,\mathbb{Z})\times\Omega^n_{cl} \mid x=[\omega]\}$

Examples: $H(n)^n(M)$ = differential characters of degree $(n-1)$ (Cheeger - Simons)

$H(2)^2(M)$ = iso classes of $U(1)$ bundles with connections

Similarly we can give the cochain model of

$$\mathrm{filt}_n\left(K(\mathbb{Z},k),\, i\right)^M$$

namely we get the truncated complex

$$\cdots \to C(n)^{k-2}(M) \to C(n)^{k-1}(M) \to C(n)^k(M),$$

6.

Let now X - topological space

$V \to X$ - vector bundle of rank d

Let $\bar{V}$ be the Thom complex of V = One point compactification of X

Consider the space

$$\mathrm{Map}(M, \bar{V})$$

of all continuous maps (with the compact - open topology)

The simplicial set

$$\mathrm{Map}(M \times \Delta^{\bullet}, \bar{V})$$

has a topological realization which is homotopy equivalent to $\mathrm{Map}(M, \bar{V})$

For practical purposes we need to look not at all maps but only at maps that are transverse to the zero section.

Pr...

Crucial remark: $\pitchfork$-Map $(M, \bar{V})$ is not in general homotopy equivalent to Map $(M, \bar{V})$. However the simplicial set $\pitchfork$-Map$(M \times \Delta^\bullet, \bar{V})$ is homotopy equivalent to Map $(M \times \Delta^\bullet, \bar{V})$

Example: Map $(pt, \mathbb{R}) = \mathbb{R}$
$\pitchfork$-Map $(pt, \mathbb{R}) = \mathbb{R} - \{0\}$

Note: $\pitchfork$-Map $(M \times \Delta^\bullet, \bar{V})$ is useful since it is homotopy equivalent to

$$\left\{ \begin{array}{l} \Sigma \subseteq M \times \Delta^k \quad \text{- codimension } k \\ \big\downarrow \leftarrow \text{classifying the normal} \\ X \quad \text{bundle} \end{array} \right\}$$

The equivalence is given by taking a map
$$f : M \times \Delta^k \to \bar{V}$$
and sending it to f^{-1}(0-section)

Differential functions at topological field theories:

Let $(X, \iota \in Z^d(X, \mathbb{Z}))$ and let

$$(c, h, \omega) : M \longrightarrow (X, \iota)$$

be a differential function with

$$(c^* \iota, h, \omega) \in Z(d)^d(M)$$

If we have a differential function

$$M^{d-1} \times S \to (X, \iota)$$

i.e. cochain element $(c, h, \omega) \in Z(d)^d(M^{d-1} \times S)$ (here $\dim S = 1$), then

$$\int_M (c^*\iota, h, \omega) \in Z(1)^1(S)$$

$$\downarrow$$

$$H(1)^1(S)$$

$$\|$$

$$e^{2\pi i \int_M h} \in \mathrm{Map}(S, U(1))$$

term in the Lagrangian in TFT

9.

Similarly if we have a differential function

$$(c, h, \omega) : M^{d-2} \times S \to (X, \iota)$$

with $\dim S = 2$

$$(c, h, \omega) \in Z^d(d)(M^{d-2} \times S)$$

$\Rightarrow$

$$\int_M (c^*\iota, h, \omega) \in Z(2)^2(S)$$

$\quad \| \quad$ U(1) bundles with connection

e.g.

$$O \hookrightarrow O \times S \to (X, \iota) \quad \mapsto \quad Z(2)^2(S \times \ldots)$$

$$d \quad \times S$$

$\| \quad$ iso classes of principal bundles.

10.

Examples: • $X = G$, $\bar{\iota} \in Z^3(X)$
$\Rightarrow$ get the Wess-Zumino term in TFT

• $X = BG$, $\bar{\iota} \in Z^4(X)$
$\Rightarrow$ get Chern-Simons term

These field theories are very linear. We would like to use the filtration structure of the differential function space to get non-linear theory.

Classical geometric story we would like to generalize:

Let M be a $Spin^c$ manifold. Then

$$(\text{index } \not{D})_4 = \frac{c^2}{8} - \frac{p_1}{24} =: \chi(c)$$

Consider a variation of the $Spin^c$ structure

$$c \mapsto c - 2x$$

where x - hermitian line bundle

$$\chi(c - 2x) - \chi(c) = \frac{x^2 - xc}{2}$$

This variation gives a universal relation between chern classes which is not quite cohomological (we have the wave sense of $x^2/2$ rather than x^2)

Looking at

$$\left(\text{index } \not{\partial}\right)_6 = \frac{c^3 - p_1 \cdot c}{48} := \chi(C)$$

(this corresponds to a 5 dimensional field theory)

Now under variation we get

$$\chi(C + dx) - \chi(x) = \frac{x^3}{6} + \frac{cx^3}{2} + \frac{3c^2 - p_1}{24} x$$

We would like to generalize this to higher dimension. For example in the story of an M-theory 5-brane we encounter a term

$$\frac{x^2 - x\lambda}{2} \qquad x - 4\text{ form}$$

(differential 4-cocycle)

and in the M-theory action we get $\frac{x^3}{6} + \ldots$

These terms have no obvious index theory interpretation (However see recent work of Diaconescu - Freed - Moore giving an interpretation in terms of E_8 index theory) So it is hard to deduce that the classes one gets by integration are integral.

We will give a topological explanation of this integrality.

---

3 dimensional topological field theory associated to pfaffian ($\not{\partial}$)

This theory is supposed to associate

$M^3 \mapsto$ point in $U(1)$

$M^2 \mapsto$ hermitian line

 $\mapsto$ map of Hermitian lines

$M_1 \amalg M_2 \mapsto \otimes$ of Hermitian lines

These data form a Picard category i.e. a category which is a:

- groupoid
- has a symmetric monoidal $\otimes$
- every object has an inverse w.r.t. $\otimes$

Picard categories are easy to classify: If $\mathcal{C}$ - Picard - category then

$\pi_0 \mathcal{C}$ = iso-classes of objects in $\mathcal{C}$

$\pi_1 \mathcal{C} = \mathrm{aut}(e)$; e - identity for $\otimes$.

Note that $\forall\, a \in \mathrm{Ob}(\mathcal{C}) \Rightarrow$

$$\mathrm{aut}(a) \cong \mathrm{aut}(e) \quad \text{- canonically}$$

indeed

$$a \xrightarrow{f} a \longmapsto \begin{array}{ccc} a\otimes a^{-1} & \xrightarrow{f\otimes 1} & a\otimes a^{-1} \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ e & \longrightarrow & e \end{array}$$

Examples:

- $\mathcal{C}$ = category of line bundles on a space
- $\mathcal{C} = \pi_{\leq 1}$ (commutative topol. group)

If now, $\mathcal{C}$ is a Picard category with

$$A = \pi_0(\mathcal{C})$$
$$B = \pi_1(\mathcal{C})$$

then C has another natural invariant

$$\forall\ a \in \mathcal{C} \Rightarrow a \otimes a \xrightarrow{\text{flip}} a \otimes a$$

gives a canonical element

$$\varepsilon \in \mathrm{aut}(e) = B$$

This gives a map

$$A \otimes \mathbb{Z}/2 \to B \overset{\text{def}}{=} \text{the k-invariant of } \mathcal{C}$$

<u>Note:</u> If $n \geq 2 \Rightarrow$

$$[K(A,2), K(B,n+2)] = \mathrm{Hom}(A \otimes \mathbb{Z}/2, B)$$
$$H^{n+2}(K(A,n), B)$$

$\Rightarrow$ the map

$$\begin{array}{c} K(A,2) \\ \downarrow \\ K(B,n+2) \end{array}$$

has some homotopy fiber X and

$\pi_n(X) = A$, $\pi_{n+1}(X) = B$, $\pi_i(X) = 0$ otherwise

So a Picard category has a natural generalization to any space X with two non-trivial consecutive homotopy groups.

Going back to $\mathcal{C}$ = Category of Hermitian lines we note that

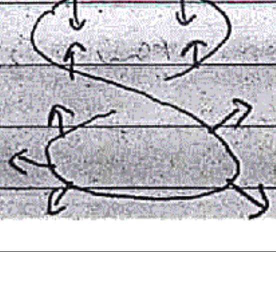
$$\pi_0 \mathcal{C} = \{e\}$$
$$\pi_1 \mathcal{C} = U(1)$$

Since $\pi_0 \mathcal{C} = \{e\}$ we can not have non-trivial K-invariant here.

However our pfaffian TFT is supposed to be a monoidal functor from the Picard category of bordisms to the Picard category $\mathcal{C}$.

But if we take Σ = pt with a non-bounding spin structure

then



gives $e^{\pi i \eta(\Sigma \times S^1)} = -1$

So we must have a target Picard category $\mathcal{C}$ same category with a non-trivial k-invariant.

Solution: Take the Picard category $\mathcal{C}$ of $\mathbb{Z}/2$ graded Hermitian lines. Here we have

$$\pi_0 \mathcal{C} = \mathbb{Z}/2$$

$$\pi_1 \mathcal{C} = U(1)$$

and a non-trivial k-invaria