Higgs bundles and D-branes - 1

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- Review of the spectral construction
- Higgs bundles
- D-branes and Higgs bundles

1. Review of spectral covers

Common strategy in mathematics: search for a duality operation that will simplify a given problem.

Example: Fourier transform for functions on a locally compact abelian group. Gives a way of converting between continuous and discrete data.

The *spectral cover construction* is another example. Rougly this is a duality operation which aims to replace a linear operator by its spectrum.

Simplest setup: Let V be a finite dimensional $\mathbb{C}$-vector space and let $\phi : V \to V$ be an endomorphism.

When ϕ is generic (=diagonalizable) one can describe ϕ via its spectral data, i.e. by giving:

- the eigenvalues of ϕ;
- the decomposition of V into a direct sum of ϕ-eigenlines;
- a mathcing between eigenvalues and eigenlines.

If $\dim_{\mathbb{C}} V = n$ this means that we are specifying n complex numbers

$$\lambda_1, \ldots, \lambda_n \in \mathbb{C} \quad (= \text{ spectrum of } \phi)$$

and to each such number we are prescribing a line $L_i \subset V$, so that $L_1 \oplus \ldots \oplus L_n = V$.

The spectral covers appear when we let this picture vary in families.

If $\phi_s : V \to V$ is a family of endomorphisms parameterized by $s \in S$, then by repeating the construction for each s we get a subvariety $\overline{S} \subset S \times \mathbb{C}$, where

$$\overline{S} = \{(s, \lambda) \,|\, \lambda \text{ is an eigenvalue of } \phi_s\}$$

If all ϕ_s have distinct eigenvalues we also get a family of eigenlines $L_{(s,\lambda)}$ parameterized by the points of $\overline{S}$.

The space $\overline{S}$ is called the *spectral cover* corresponding to the family $\{\phi_s\}_{s \in S}$. Under the genericity assumption $\overline{S}$ is an unramified n-sheeted cover of S and it carries a line bundle consisting of all eigenvalues of the ϕ_s's.

Note: The data $(\overline{S} \to S, L \to \overline{S})$ completely reconstructs the family $\{\phi_s\}_{s \in S}$.

The correspondence $\{\phi_s\}_{s \in S} \leftrightarrow (\overline{S}, L)$ is not very useful under the genericity assumption. In applications one needs to deal with ϕ_s which have repeated eigenvalues. In this case $\overline{S} \to S$ becomes ramified over $s \in S$ and the fibers of $L \to \overline{S}$ may jump at the multiple valued points.

Thus one expects some kind of a sheaf structure for L along the ramification locus of $\overline{S} \to S$.

Important special case: Allow ϕ_s to have multiple eigenvalues but require that there is exactly one Jordan block per eigenvalue.

Such an endomorphism of V is called regular. It carries a single eigenline per eigenvalue. In particular if all ϕ_s, $s \in S$ are regular we get again a line bundle $L \to \overline{S}$ on the spectral cover $\overline{S}$.

More invariantly, consider the polynomial map

$$h: \quad \mathrm{End}(V) \longrightarrow \mathbb{C}^n$$
$$\phi \longmapsto (a_1(\phi), \ldots, a_n(\phi)),$$

where the $a_i(\phi)$'s are the coefficients

$$\det(t \cdot \mathrm{id}_V - \phi) = t^n + a_1(\phi)t^{n-1} + \ldots + a_n(\phi).$$

of the characteristic polynomial of ϕ.

The spectrum of ϕ depends only on $h(\phi)$ and so $\overline{S}$ is just the pullback via the map

$$S \longrightarrow \mathbb{C}^n$$
$$\phi_s \longrightarrow h(\phi_s).$$

of the obvious cover

$$\overline{\mathbb{C}}^n \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n,$$

given by the equation $t^n + a_1 t^{n-1} + \ldots + a_n = 0$ in the coordinates $(a_1, \ldots, a_n; t) \in \mathbb{C}^n \times \mathbb{C}$.

The fibers of h : End$(V) \to \mathbb{C}^n$ are invariant under conjugation action of $GL(V)$ and in fact

$$\mathbb{C}^n = \text{End}(V)//GL(V)$$

is the GIT quotient.

Explanation: The orbits of the $GL(V)$-action on End(V) are not all closed and so the natural topology on the set of orbits End$(V)/GL(V)$ will not be Hausdorff. To remedy that one looks for a space End$(V)//GL(V)$ parameterizing the closures of $GL(V)$-orbits in End(V).

For a general (regular and semisimple) ϕ in End(V) the $GL(V)$-orbit is closed and in a neighborhood of such ϕ the quotients

$$\text{End}(V)/GL(V) \quad \text{and} \quad \text{End}(V)//GL(V)$$

coincide.

When ϕ is arbitrary, then $\overline{GL(V)\cdot\phi}$ contains a unique closed and a unique open orbit. The closed one is the orbit of a semisimple (diagonalizable) endomorphism and the open one is the orbit of a regular endomorphism. This leads to

Two interpretations for End$(V)//GL(V)$*:* either as the space parameterizing semisimple endomorphisms modulo conjugation, or as the space parameterizing all regular endomorphisms modulo conjugation.

Both interpreations are useful but the one for which the eigenlines vary 'continuosly' is the interpretation via regular endomorphisms.

Example: Let $\dim_{\mathbb{C}}(V) = 2$. Use $SL(V)$ instead of $GL(V)$. Then we have $h : SL(V) \to \mathbb{C}$, $h(\phi) = \det\phi$, and if $\det\phi \neq 0$, then ϕ is regular and semisimple. If $\det\phi = 0$, then ϕ is nilpotent and then

$$h^{-1}(0) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \coprod \left\{ SL(V)\cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

Extensions and generalizations

To make the simple-minded spectral cover construction useful in applications one needs to extend it in two ways:

(i) Allow for abitrary groups G, not only for $GL(V)$;
(ii) Allow for twisted versions of ϕ.

For **(i)**: Instead of looking at elements $\phi \in \mathrm{End}(V)$ we take elements $\phi \in \mathfrak{g}$ for $\mathfrak{g} :=$ Lie(G) of some complex semi-simple group G. The spectral cover construction in this case is somewhat subtler since it has to reflect the complexity of the group G. I will not discuss this part of the story. For more details see Ron Donagi's papers . . . .

For **(ii)**: The 'twisting' of the $\{\phi_s\}_{s\in S}$ can be achieved in two ways. Firstly, one can allow for the vector space V to vary with the point $s \in S$. This is easily realized by replacing $S \times V$ by a non-trivial vector bundle E on S. In this setup the family of endomorphisms naturally should be viewed as a section $\phi \in \Gamma(S, \mathrm{End}(E))$. Secondly, one can allow for ϕ to have nontrivial coefficients in some coefficient object K.

The freedom of choosing K is essential in the applications. Since the elements in K can be thought of as the matrix coefficients of ϕ, it is natural to require that K has an abelian group structure. Possible natural choices for K are: a vector bundle on X, a family of affine tori on X, a family of abelian varieties on X or more generally a family of commutative group stacks over X.

We will see examples of most of these choices later on and will relate them to D-brane moduli and dualities. The simplest choice is to take K to be a vector bundle. This leads to the classical notion of a Higgs bundle.

2. Higgs bundles

Let S be a a complex algebraic variety and let K be a fixed algebraic vector bundle of rank n on S. Consider a vector bundle $E \to S$ of rank r and an $\mathcal{O}_S$-linear map

$$\phi : E \to E \otimes K.$$

We would like to take the 'spectrum' of ϕ and recast the data (E, ϕ) in terms of a spectral cover C of S possibly decorated with some additional structure (e.g. a coherent sheaf).

Problem The spectrum may not be well defined for a general ϕ.

Indeed, if we trivialize K locally on S, i.e. if we choose a local frame $K_{|V} \cong \mathbb{C}^n \otimes \mathcal{O}_V$, then we see that locally ϕ comprises n endomorphims

$$\phi_{|V} = (\phi_1, \ldots, \phi_n), \text{ with } \phi_i \in \Gamma(V, \mathrm{End}(E)).$$

We can apply the naive spectral construction to each ϕ_i but the collection of spectral covers we will get this way will depend on the trivialization of K.

To fix that one may look only at ϕ's for which all the ϕ_i's behave in the same way e.g. are simultaneously diagonalizable. More generally we can require that $[\phi_i, \phi_j] = 0$ for all i, j, i.e. that the ϕ_i's generate a commutative subalgebra in End(E). The latter condition is clearly equivalent to requiring that

$$\phi \wedge \phi = 0 \in \Gamma(S, \mathrm{End}(E) \otimes \bigwedge^2 K).$$

This motivates the following

Definition *A K-valued Higgs bundle on an algebraic variety S is a pair*

$$(E, \phi : E \to E \otimes K)$$

satisfying $\phi \wedge \phi = 0$.

Similarly one defines a Higgs coherent sheaf on S.

Observe that for a Higgs bundle (E, ϕ), the Higgs field ϕ can be interpreted as a map $K^\vee \otimes E \to E$ and so generates an action $\boldsymbol{T}K^\vee \otimes E \to E$ of the sheaf of tensor algebras $\boldsymbol{T}K^\vee := \oplus_i (K^\vee)^{\otimes i}$ on E. The condition $\phi \wedge \phi = 0$ is equivalent to saying that this action descends to an action

$$S^\bullet K^\vee \otimes E \to E$$

of the symmetric algebra $S^\bullet K^\vee$ on E.

This fact admits a geometric interpretation. Consider the total space $X := \mathrm{tot}(K)$ of the vector bundle K. Let $p : X \to S$ be the natural projection. Then p is an affine map and

$$p_* \mathcal{O}_X = S^\bullet K^\vee, \qquad X = \mathrm{Spec}(S^\bullet K^\vee).$$

In particular, a quasi-coherent sheaf $\mathscr{E}$ on X is the same thing as a quasi-coherent sheaf $E (= p_* \mathscr{E})$ on S together with a $S^\bullet K^\vee$-action.

Note that since p is affine, an $S^\bullet K^\vee$-module E which is coherent as a sheaf on S will correspond to a coherent sheaf $\mathscr{E}$ on X which is *finite* over S. In other words we have an equivalence of categories

$$p_* : \left\{ \begin{matrix} \text{quasi-coherent} \\ \text{sheaves on } X \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{Sheaves of } S^\bullet K^\vee \\ \text{modules on } S\text{, quasi-} \\ \text{coherent as sheaves of} \\ \mathcal{O}_S\text{-modules} \end{matrix} \right\}$$

which restricts to an equivalence

$$p_* : \left\{ \begin{matrix} \text{coherent sheaves} \\ \text{on } X\text{, finite over } S \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{Higgs coherent} \\ \text{sheaves on } S \end{matrix} \right\}.$$

This is the *K-valued spectral correspondence*. It converts spectral data (= coherent sheaves on tot(K) whose support is finite over S) to Higgs data (= K-twisted families of endomorphisms on S).

Remark: • The Higgs sheaf (E, ϕ) corresponding to a sheaf $\mathscr{E}$ on X can be described explicitly:

$E = p_*\mathscr{E}$ is the pushforward of $\mathscr{E}$,

$\phi : E \to E \otimes K$ is the pushforward of $\mathscr{E} \overset{\otimes\lambda}{\to} \mathscr{E} \otimes p^*K$ where, $\lambda \in \Gamma(\text{tot}(K), p^*K)$ is the tautological section.

• If a sheaf $\mathscr{E}$ on X corresponds to a Higgs bundle (E, ϕ) of rank r, then the spectral cover for (E, ϕ) is defined as the subscheme Supp$(\mathscr{E}) \subset X$ which maps onto S and is finite of degree r over S. It is given explicitly as the zero locus of the section

$$\det(\lambda \cdot \text{id} - p^*\phi) \in \Gamma(X, p^*S^rK).$$

• When K is the trivial line bundle on S, then $X = S \times \mathbb{C}$ and we recover the old definition of a spectral cover for a family of endomorphisms.

3. D-branes and Higgs bundles

The spectral correspondence is a simple geometric duality which can be used to describe D-branes.

Setup 1: Let (S, g) be a compact Kähler manifold with g real-analytic, $K = \Omega^1_S$ the holomorphic cotangent bundle of S. The total space $X = \text{tot}(K)$ of K carries a holomorphic symplectic form - the exterior derivative $\Omega = d\lambda$ of the tautological one form λ on X. It is known (B.Feix'99, D.Kaledin'99) that a tubular neighborhood of the zero section $S \subset X$ of K supports a unique hyper-Kähler metric which is compatible with Ω and restricts to g on S. Thus X is a non-compact, non-complete physicists Calabi-Yau manifold which can be taken as a string background.

The B-branes on X are coherent sheaves on X with compact support, i.e. coherent sheaves $\mathscr{E}$ on X which are finite over S. By the spectral correspondence one can describe the moduli space of such $\mathscr{E}$ as the moduli space of Higgs bundles $(E, \phi : E \to E \otimes \Omega^1_S)$ on S. If the Chern classes of $\mathscr{E}$ are chosen so that $c_1(E) = 0$ and $c_2(E) = 0$, then all such Higgs bundles correspond to representations of $\pi_1(S)$ by C.Simpson's theory.

This gives a concrete description of a component of the moduli space of B-branes on X.

Setup 2: Let Z be a three dimensional compact Calabi-Yau manifold. Let $C \subset Z$ be a smooth rigid curve in Z. In M-theory one is interested in the moduli space BPS(Z, C, r) of BPS states on Z of charge $r \cdot [C] \in H_2(Z, \mathbb{Z})$. Geometrically BPS$(Z, C, r)$ should parameterize torsion sheaves on Z whose support represents the homology class $r \cdot [C]$.

Note: This problem is not very well posed - the corresponding quot scheme is not of finite type.

On the other hand Gopakumar-Vafa gave an explicit formula (see hep-th/9812127) expressing the Euler characteristic of the space BPS(Z, C, r) in terms of finitely many GW invariants in the homology class $r \cdot [C]$.

Question: What is this formula really calculating?

One possible answer suggested by the perturbative nature of the Gopakumar-Vafa calculation is that in fact the space BPS(Z, C, r) should be linearized in a suitable way before we start counting.

Proposal: Replace the global Calabi-Yau Z by its linearization near C, i.e. by the local Calabi-Yau

$$X := \mathrm{tot}(N_{C/Z}),$$

with C sitting inside X as the zero section.

Now we have a natural projection $p : X \to C$ and the space BPS(X, C, r) is just the moduli space of coherent sheaves on X, which are finite and of degree r over C. By the spectral correspondence we can identify BPS(X, C, r) with the moduli of rank r Higgs bundles $(E, \phi : E \to E \otimes N_{C/Z})$ on C which is much simpler. In particular, one can utilize the natural $\mathbb{C}^{\times}$-action on Higgs bundles:

$$t \cdot (E, \phi) := (E, t\phi), \text{ for all } t \in \mathbb{C}^{\times},$$

and try to localize the calculation of the Euler characteristic of BPS(X, C, r) at the fixed locus of $\mathbb{C}^{\times}$.