Towards a New Algorithm for Multiphase Lattice Boltzmann Simulations

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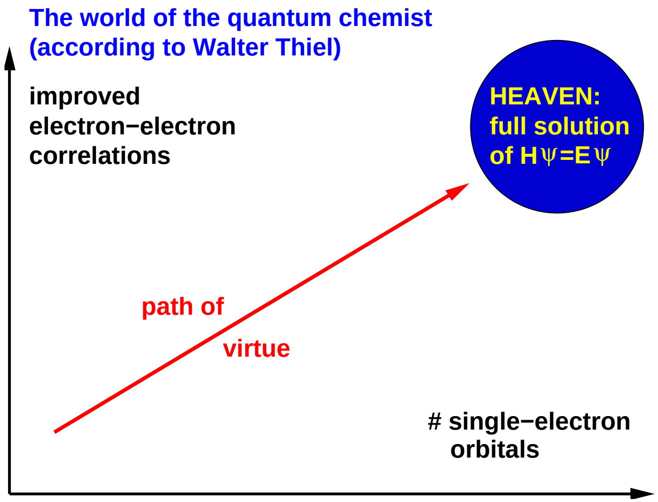
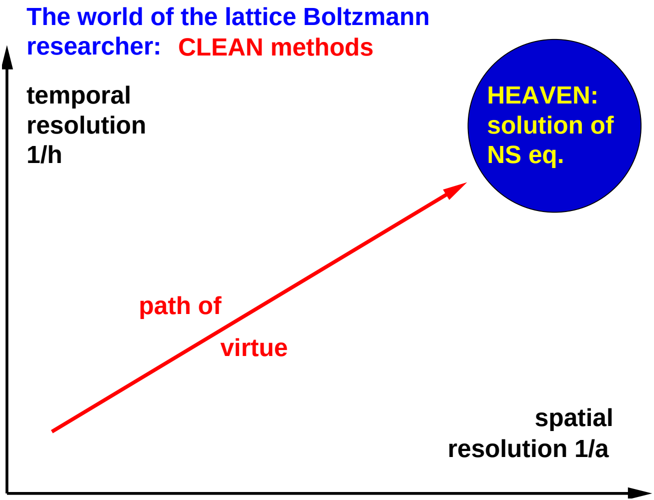
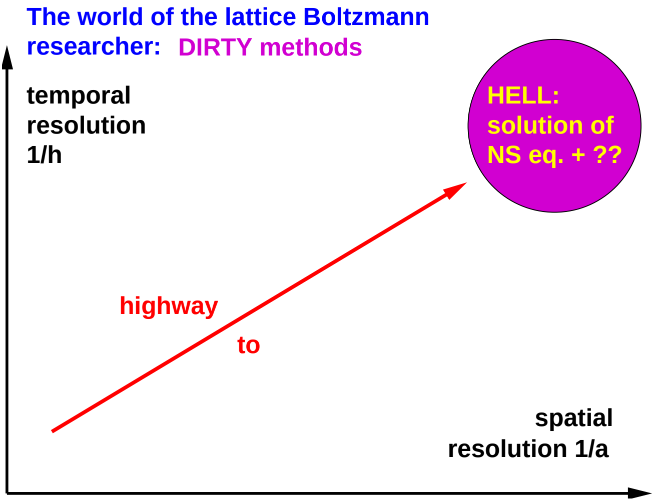
Some theology



Some theology



Some theology

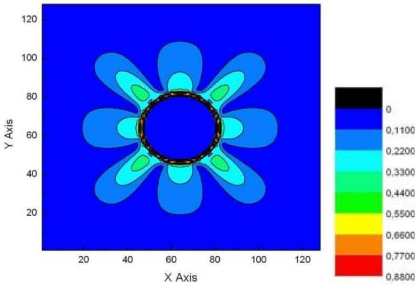


In short:

- system: single-component fluid, coexistence of gas and liquid phase
- objectives: develop thermodynamically and hydrodynamically fully consistent lattice Boltzmann algorithm for multiphase fluids
- hope / expectation: complete elimination of spurious currents

Droplet formation →

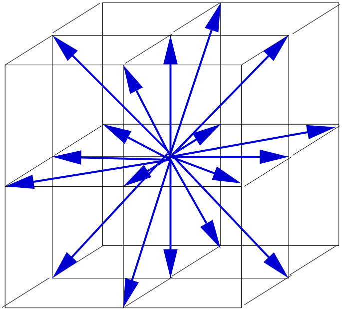
spurious currents: $\frac{|\vec{u}|}{c_s}$



A. Kuzmin et al., Int. J. Mod. Phys. C, 2008

Background and motivation:
Lattice Boltzmann for the ideal gas

Lattice Boltzmann



- linearized Boltzmann equation (kinetic theory of gases)
- fully discretized
- sites $\vec{r}$, lattice spacing a
- time t, time step h

- $\vec{c}_i$ small set of velocities
- $\vec{c}_i h$ connects two sites
- $n_i(\vec{r}, t)$: real number, mass density on site $\vec{r}$ corresponding to velocity $\vec{c}_i$

$$n_i(\vec{r} + \vec{c}_i h, t + h) = n_i^\star(\vec{r}, t) = n_i(\vec{r}, t) + \Delta_i(\vec{r}, t)$$

Conservation laws, symmetries

$$n_i(\vec{r} + \vec{c}_i h, t + h) = n_i^\star(\vec{r}, t) = n_i(\vec{r}, t) + \Delta_i \left\{ n_i(\vec{r}, t) \right\}$$

$$\rho = \sum_i n_i$$

$$\vec{j} = \rho \vec{u} = \sum_i n_i \vec{c}_i$$

$$\sum_i \Delta_i = \sum_i \Delta_i \vec{c}_i = 0$$

- 👍 mass conservation
- 👍 momentum conservation
- 👍 locality
- 👎 rotational symmetry (lattice!)
- 👎 Galilei invariance (finite number of velocities)

Low Mach number physics

- only $u \ll c_i$
- only $u \ll c_s$
- $Ma = u/c_s \ll 1$
- low Mach number $\Rightarrow$ compressibility does not matter $\Rightarrow$ equation of state does not matter $\Rightarrow$ choose ideal gas!

m_p particle mass:

$$p = \frac{\rho}{m_p} k_B T$$

$$c_s^2 = \frac{\partial p}{\partial \rho} = \frac{1}{m_p} k_B T$$

$$p = \rho c_s^2$$

$$k_B T = m_p c_s^2$$

Desired asymptotic limit: Navier–Stokes

in the continuum limit $a \to 0$, $h \to 0$ (or: on large length and time scales)

$$\partial_t \rho + \partial_\alpha j_\alpha = 0$$

$$\partial_t j_\alpha + \partial_\beta \left(\rho c_s^2 \delta_{\alpha\beta} + \rho u_\alpha u_\beta \right) = \partial_\beta \sigma_{\alpha\beta}$$

$$\sigma_{\alpha\beta} = \eta_{\alpha\beta\gamma\delta} \partial_\gamma u_\delta$$

$$\eta_{\alpha\beta\gamma\delta} = \left(\zeta - \frac{2}{3}\eta \right) \delta_{\alpha\beta}\delta_{\gamma\delta} + \eta \left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} \right)$$

- η shear viscosity
- ζ bulk viscosity

LB continuum limit: how??

in principle, only one parameter $\varepsilon \to 0$:

- wave–like scaling: $a/h = const.$

$$\begin{aligned} a &= \varepsilon a_0 \\ h &= \varepsilon h_0 \\ \vec{c}_i &= \vec{c}_{i0} \\ n_i(\vec{r} + \varepsilon \vec{c}_{i0} h_0, t + \varepsilon h_0) - n_i(\vec{r}, t) &= \Delta_i \end{aligned}$$

- diffusive scaling: $a^2/h = const.$

$$\begin{aligned} a &= \varepsilon a_0 \\ h &= \varepsilon^2 h_0 \\ \vec{c}_i &= \varepsilon^{-1} \vec{c}_{i0} \\ n_i(\vec{r} + \varepsilon \vec{c}_{i0} h_0, t + \varepsilon^2 h_0) - n_i(\vec{r}, t) &= \Delta_i \end{aligned}$$

The idea of multiple time scale expansion

example:

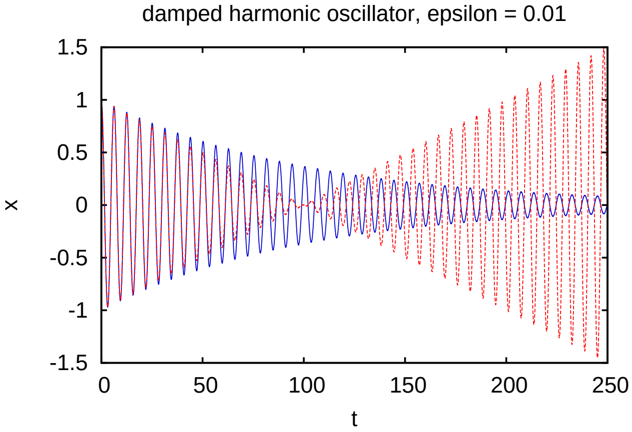
- weakly damped harmonic oscillator
- $\varepsilon \to 0$
- time scale separation: damping time $\gg$ oscillation time

$$\frac{d^2}{dt^2}x + 2\varepsilon\frac{d}{dt}x + x = 0$$

exact solution:

$$x(t) = \exp\left(-\varepsilon t\right)\cos\left(\sqrt{1-\varepsilon^2}\,t\right)$$

naive Taylor expansion wrt ε (1st order):



Multiple time scale analysis

Idea:

$$\begin{aligned} x(t) &= \exp(-\varepsilon t)\cos\left(\sqrt{1-\varepsilon^2}\,t\right) \\ &\approx \exp(-\varepsilon t)\cos t \\ &= \exp(-t_2)\cos t_1 \\ &= x(t_1, t_2) \end{aligned}$$

with

$$t_1 = t \qquad t_2 = \varepsilon t$$

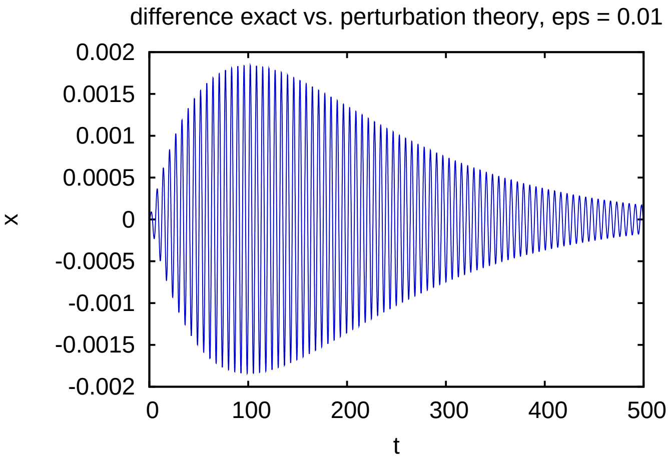
consider x as a function of two independent variables t_1, t_2
$\Rightarrow$ should be able to grasp the time scale separation!
hence, study expansion

$$x(t_1, t_2) = x^{(0)}(t_1, t_2) + \varepsilon x^{(1)}(t_1, t_2) + \varepsilon^2 x^{(2)}(t_1, t_2) + \ldots$$

with

$$\frac{d}{dt} = \frac{\partial t_1}{\partial t}\frac{\partial}{\partial t_1} + \frac{\partial t_2}{\partial t}\frac{\partial}{\partial t_2} = \frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2}$$

result for leading order:



Chapman–Enskog expansion

- LBE:
$$n_i(\vec{r} + \vec{c}_i h, t + h) - n_i(\vec{r}, t) = \Delta_i$$
- $$D_i = c_{i\alpha} h \partial_\alpha + h \partial_t$$
- exact re–write as a differential equation:
$$D_i n_i = \left[1 - \frac{D_i}{2} + \frac{D_i^2}{12} + \dots\right] \Delta_i$$
- $\vec{r}_1 = \varepsilon \vec{r}$
- $t_1 = \varepsilon t$ waves
- $t_2 = \varepsilon^2 t$ diffusion
- $n_i = n_i(\vec{r}_1, t_1, t_2)$
- $$\partial_\alpha = \varepsilon \partial_{1\alpha}$$
- $$\partial_t = \varepsilon \partial_{t1} + \varepsilon^2 \partial_{t2}$$

Expanding the solution

$$
\begin{aligned}
n_i &= n_i^{(0)} + \varepsilon n_i^{(1)} + O(\varepsilon^2) \\
\Delta_i \{n_i\} &= \Delta_i \left\{ n_i^{(0)} \right\} + \varepsilon \sum_j L_{ij} n_j^{(1)} + O(\varepsilon^2) \\
&= \Delta_i \left\{ n_i^{(0)} \right\} - \varepsilon (1-\gamma) n_i^{(1)} + O(\varepsilon^2) \quad \text{(BGK)}
\end{aligned}
$$

ε orders

- ε^0: $n_i^{(0)}$ is collisional invariant, hence $n_i^{(0)} = n_i^{eq} = n_i^{eq}(\rho, \vec{j})$
- $n_i^{(0)}$ contains NO GRADIENTS! THESE APPEAR AT HIGHER ORDER!
- ε^1: Continuity equation, dissipation–free NS
- isotropic NS requires

$$n_i^{eq}(\rho, \vec{u}) = w_i \rho \left(1 + \frac{\vec{u} \cdot \vec{c}_i}{c_s^2} + \frac{(\vec{u} \cdot \vec{c}_i)^2}{2c_s^4} - \frac{u^2}{2c_s^2} \right)$$

- 3 shells needed. D3Q19: $w_0 = 1/3$, $w_1 = 1/18$, $w_2 = 1/36$, $c_s^2 = a^2/(3h^2)$
- more freedom in the equation of state requires more shells
- ε^2: Dissipative part of NS
-
$$\eta = \frac{h}{2} \rho c_s^2 \frac{1 + \gamma}{1 - \gamma}$$

Gas–liquid systems, part 1:

Lattice Boltzmann setup and
the difference between ideal gas and nonideal gas

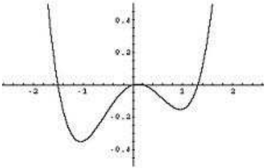
Desired limit: Model H

- Starting point: Landau–Ginzburg free energy functional
- In the absence of dissipation, this should be viewed as the conserved energy:

$$\mathcal{H} = \int_V dV \left[\frac{1}{2}\rho\vec{u}^2 + \rho e + \frac{\kappa}{2}(\nabla\rho)^2 \right]$$

$e(\rho)$ gives rise to a Maxwell loop in $p(\rho)$

e vs. ρ



Conservative dynamics:

$$\partial_t \rho + \partial_\alpha j_\alpha = 0$$
$$\frac{\partial}{\partial t}(\rho u_\alpha) + \partial_\beta(\rho u_\alpha u_\beta) = -\partial_\alpha p + f_\alpha$$

Choose force such that Hamiltonian is conserved:

$$\boxed{\frac{d}{dt}\mathcal{H} = 0} \rightarrow \int dV\, u_\alpha f_\alpha = -\frac{\kappa}{2}\int dV \frac{\partial}{\partial t}\left(\nabla\rho\right)^2$$

$\Rightarrow$

$$\boxed{f_\alpha^{int} = \kappa\rho\partial_\alpha\partial_\beta\partial_\beta\rho} \quad \leftarrow \quad \text{3rd order gradient!}$$

$\Rightarrow$ Chapman–Enskog up to 3rd order is needed!
needed in LB:

$$\Delta_i = \Delta_i^{bulk} + \Delta_i^{int} + \Delta_i^{corr} = -(1-\gamma)(n_i - n_i^{eq}) + \Delta_i^{int} + \Delta_i^{corr}$$

Equilibrium populations

$$n_i^{eq}(\rho, \vec{u}) = w_i \rho \left(1 + \frac{\vec{u} \cdot \vec{c}_i}{c_s^2} + \frac{(\vec{u} \cdot \vec{c}_i)^2}{2c_s^4} - \frac{u^2}{2c_s^2} \right)$$

Equation of state

Ideal gas:

$$p = \rho c_s^2 \to c_s^2 = \frac{p}{\rho} = \frac{\partial p}{\partial \rho}$$

Nonideal gas:

$$p(\rho) = \rho c_s^2(\rho) \to c_s^2(\rho) = \frac{p}{\rho} \neq \frac{\partial p}{\partial \rho}$$

Velocity moments

$$\sum_i n_i^{eq} = \rho$$

$$\sum_i n_i^{eq} c_{i\alpha} = j_\alpha$$

$$\sum_i n_i^{eq} c_{i\alpha} c_{i\beta} = \pi_{\alpha\beta}^{(eq)}$$

$$= p(\rho)\delta_{\alpha\beta} + \rho u_\alpha u_\beta$$

Non-equil. moments: defined similarly up to 4th order.

Coefficients - IDEAL GAS

isotropic lattice tensors

$$\sum_i w_i c_{i\alpha} = 0$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} = c_s^2 \quad \delta_{\alpha\beta}$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} c_{i\gamma} = 0$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} c_{i\gamma} c_{i\delta}$$

$$= c_s^4 \quad (\delta_{\alpha\beta}\delta_{\gamma\delta} + \text{perm.})$$

weights

D3Q19: Qian et al., EPL, 1992

(000) $w_0 = \frac{1}{3}$

(100) $w_1 = \frac{1}{18}$

(110) $w_2 = \frac{1}{36}$

- $c_s^2 = 1/3$

in lattice units

Coefficients - NONIDEAL GAS

isotropic lattice tensors

$$\sum_i w_i c_{i\alpha} = 0$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} = c_s^2(\rho)\delta_{\alpha\beta}$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} c_{i\gamma} = 0$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} c_{i\gamma} c_{i\delta}$$

$$= c_s^4(\rho)(\delta_{\alpha\beta}\delta_{\gamma\delta} + \text{perm.})$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} c_{i\gamma} c_{i\delta} c_{i\mu} = 0$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} c_{i\gamma} c_{i\delta} c_{i\mu} c_{i\nu} =$$

$$c_s^6(\rho)(\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\mu\nu} + \text{perm.})$$

weights

D3Q19: Qian et al., EPL, 1992

(000) $w_0 = \frac{1}{3}$

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Coefficients - NONIDEAL GAS

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$$\sum_i w_i c_{i\alpha} = 0$$

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$$\sum_i w_i c_{i\alpha} c_{i\beta} c_{i\gamma} = 0$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} c_{i\gamma} c_{i\delta} = c_s^4(\rho)(\delta_{\alpha\beta}\delta_{\gamma\delta} + \text{perm.})$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} c_{i\gamma} c_{i\delta} c_{i\mu} = 0$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} c_{i\gamma} c_{i\delta} c_{i\mu} c_{i\nu} = c_s^6(\rho)(\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\mu\nu} + \text{perm.})$$

weights

D3Q59: H. Chen et al., J. Sci. Comp., 2008

(000) $w_0 = 1 - \frac{2351}{720} c_s^2 + \frac{1081}{192} c_s^4 - \frac{363}{192} c_s^6$

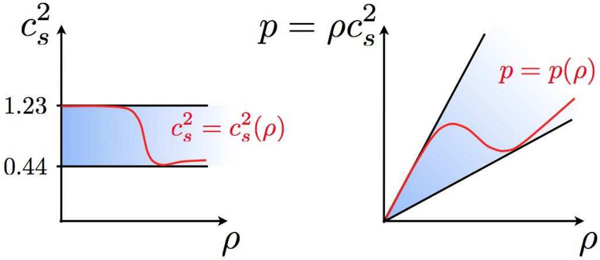
(100) $w_1 = 2c_s^2(\frac{8}{45} - \frac{1}{3} c_s^2 + \frac{1}{6} c_s^4)$

(110) $w_2 = c_s^2(\frac{8}{45} - \frac{1}{3} c_s^2 + \frac{1}{6} c_s^4)$

- $c_s^2 \in [0.44, 1.23]$ $(w_i > 0!)$

in lattice units
further weights $w_3, \ldots, w_7$ for
(111), (200), (220), (222), (400)

Allowed equation of state



Comparison to Swift-Yeomans

Swift-Yeomans

- determine pressure tensor: $P^{int}_{\alpha\beta}$
- impose: $\sum_i n_i^{eq} c_{i\alpha} c_{i\beta} = p\delta_{\alpha\beta} + \rho u_\alpha u_\beta + P^{int}_{\alpha\beta}$

Problem with S-Y model:

- $n_i^{eq} \to$ 0th order CE
- $P^{int}_{\alpha\beta} \to$ higher order CE because of gradients!

This work

- determine force density: f^{int}_α
- put it into the collision operator: $\Delta_i^{int} = w_i \frac{2h}{1+\gamma} \frac{f^{int}_\alpha c_{i\alpha}}{c_s^2}$

Decouple bulk and interface terms:

- bulk $\leftarrow$ local equilibrium populations
- interface $\leftarrow$ force

Gas–liquid systems, part 2:
Chapman-Enskog analysis

Chapman–Enskog expansion

$$
\begin{aligned}
\vec{r}_1 &= \varepsilon \vec{r} && \text{scaling parameter } \varepsilon \\
t_1 &= \varepsilon t && \text{sound waves} \\
t_2 &= \varepsilon^2 t && \text{diffusion of momentum} \\
t_3 &= \varepsilon^3 t && \text{coarsening}
\end{aligned}
$$

$$n_i(\vec{r}_1 + \vec{c}_i \varepsilon h, t_1 + \varepsilon h, t_2 + \varepsilon^2 h, t_3 + \varepsilon^3 h) - n_i(\vec{r}_1, t_1, t_2, t_3) = \Delta_i$$

From Boltzmann to Navier–Stokes

Lattice Boltzmann eq.:

$$n_i(\vec{r}_1 + \varepsilon \vec{c}_i h, t_1 + \varepsilon h, t_2 + \varepsilon^2 h, t_3 + \varepsilon^3 h) - n_i(\vec{r}_1, t_1, t_2, t_3)$$
$$= \Delta_i^{bulk} + \Delta_i^{int} + \Delta_i^{corr}$$

Navier–Stokes eq.:

$$\partial_t j_\alpha + \partial_\beta \left(p(\rho)\delta_{\alpha\beta} + \rho u_\alpha u_\beta \right) - \partial_\beta \sigma_{\alpha\beta} = \kappa\rho\partial_\alpha\partial_\beta\partial_\beta\rho$$

Velocity moments

$$hf_\alpha = \frac{2}{1+\gamma} \sum_i \Delta_i^{int} c_{i\alpha}$$

Interface collision operator Δ_i^{int}

local collision operator - ideal gas:

$$\Delta_i(\vec{r}, t) = \Delta_i(\{n_i(\vec{r}, t)\} \qquad)$$

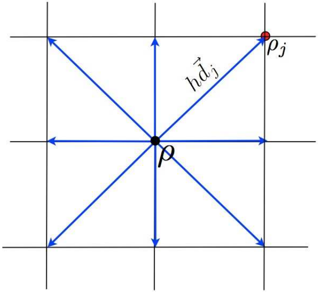
$$\rho = \rho(\vec{r}_1)$$

Interface collision operator Δ_i^{int}

Nonlocal collision operator - nonideal gas:

$$\Delta_i(\vec{r}, t) = \Delta_i(\{n_i(\vec{r}, t)\}, \{\rho_j\})$$

$$\rho = \rho(\vec{r}_1) \qquad \rho_j \equiv \rho(\vec{r}_1 + \vec{d}_j \varepsilon h)$$



- new set of lattice vectors $\vec{d}_j$
- new set of weights τ_j
- interactions with neighboring sites

Adjust weights τ_i so that:

$$\sum_i \tau_i d_{i\alpha} = 0 \qquad \sum_i \tau_i d_{i\alpha} d_{i\beta} d_{i\gamma} = 0$$

$$\sum_i \tau_i d_{i\alpha} d_{i\beta} = \widetilde{\sigma}_2 \delta_{\alpha\beta} = 0$$

$$\sum_i \tau_i d_{i\alpha} d_{i\beta} d_{i\gamma} d_{i\delta} = \widetilde{\sigma}_4 (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$$

$\Rightarrow$ 3 shells of lattice vectors $\vec{d}_j$ needed

Ansatz:

$$\Delta_i^{int} = \kappa \rho \, w_i c_{i\alpha} \sum_j \tau_j d_{j\alpha} \rho_j$$

$$\Delta_i^{int} = \kappa\rho\, w_i c_{i\alpha} \sum_j \tau_j d_{j\alpha} \rho_j$$

Does this Ansatz give us the right force?

$$\begin{aligned} \frac{1+\gamma}{2} h f_\alpha^{int} &= \sum_i \Delta_i^{int} c_{i\alpha} \\ &= \kappa\rho\, c_s^2 \sum_j \tau_j d_{j\alpha} \rho_j \end{aligned}$$

$$\Delta_i^{int} = \kappa\rho\, w_i c_{i\alpha} \sum_j \tau_j d_{j\alpha} \rho_j$$

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$$\rho_j = \rho + \varepsilon h d_{j\alpha}\, \partial_{\alpha_1} \rho + \frac{\varepsilon^2 h^2}{2} d_{j\alpha} d_{j\beta}\, \partial_{\alpha_1} \partial_{\beta_1} \rho + \frac{\varepsilon^3 h^3}{6} d_{j\alpha} d_{j\beta} d_{j\gamma}\, \partial_{\alpha_1} \partial_{\beta_1} \partial_{\gamma_1} \rho$$

$$\widetilde{\sigma}_2 = 0$$

$$\Delta_i^{int} = \kappa\rho\, w_i c_{i\alpha} \sum_j \tau_j d_{j\alpha} \rho_j$$

Does this Ansatz give us the right force?

$$\begin{aligned}
\frac{1+\gamma}{2} h f_\alpha^{int} &= \sum_i \Delta_i^{int} c_{i\alpha} \\
&= \kappa\rho\, c_s^2 \sum_j \tau_j d_{j\alpha} \rho_j \\
&= \kappa\rho c_s^2 \widetilde{\sigma}_4 h^3 \frac{1}{6}\, 3\, \partial_\alpha \partial_\beta \partial_\beta \rho
\end{aligned}$$

$$\Delta_i^{int} = \kappa\rho\, w_i c_{i\alpha} \sum_j \tau_j d_{j\alpha} \rho_j$$

Does this Ansatz give us the right force?

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\frac{1+\gamma}{2} h f_\alpha^{int} &= \sum_i \Delta_i^{int} c_{i\alpha} \\
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&= \kappa\rho c_s^2 \widetilde{\sigma}_4 h^3 \frac{1}{6}\, 3\, \partial_\alpha \partial_\beta \partial_\beta \rho
\end{aligned}$$

- $f_\alpha^{int} = \kappa\rho \partial_\alpha \partial_\beta \partial_\beta \rho$

$$\Delta_i^{int} = \kappa\rho\, w_i c_{i\alpha} \sum_j \tau_j d_{j\alpha} \rho_j$$

Does this Ansatz give us the right force?

$$\begin{aligned}
\frac{1+\gamma}{2} h f_\alpha^{int} &= \sum_i \Delta_i^{int} c_{i\alpha} \\
&= \kappa\rho\, c_s^2 \sum_j \tau_j d_{j\alpha} \rho_j \\
&= \kappa\rho c_s^2 \widetilde{\sigma}_4 h^3 \frac{1}{6}\, 3\, \partial_\alpha \partial_\beta \partial_\beta \rho
\end{aligned}$$

- $f_\alpha^{int} = \kappa\rho\partial_\alpha\partial_\beta\partial_\beta\rho$
- $\Rightarrow \widetilde{\sigma}_4 = \frac{1+\gamma}{h^2 c_s^2}$

Continuity equation

$$\boxed{\partial_t \rho + \partial_\alpha j_\alpha = 0}$$

In order to get it correct up to 3rd order in the CE expansion, we need a

Re-defined momentum density

$$j_\alpha = \sum_i n_i c_{i\alpha} + \frac{h}{2} f_\alpha - \varepsilon^2 \frac{h}{12} \partial_{\beta_1} (\pi^{*(1)}_{\alpha\beta} - \pi^{(1)}_{\alpha\beta})$$

$$\frac{1}{h}(\pi^{*(1)}_{\alpha\beta} - \pi^{(1)}_{\alpha\beta}) = p(\rho)(\partial_{\alpha_1} u_\beta + \partial_{\beta_1} u_\alpha) + \left(p(\rho) - \frac{\partial p(\rho)}{\partial \rho} \rho \right) (\partial_{\gamma_1} u_\gamma) \delta_{\alpha\beta}$$

Need an iterative algorithm to define $u_\alpha = j_\alpha / \rho$!

Navier-Stokes equation

$$
\begin{aligned}
&\partial_t j_\alpha + \partial_\beta \pi^{(0)}_{\alpha\beta} + \varepsilon \frac{1}{2}\frac{\gamma+1}{\gamma-1}\partial_\beta \left(\pi^{*(1)}_{\alpha\beta} - \pi^{(1)}_{\alpha\beta}\right) - \varepsilon \frac{h}{\gamma-1}\partial_\beta \Sigma^{(1)}_{\alpha\beta} \\
&+ \varepsilon^2 \frac{h}{2}\partial_\beta \left[\frac{\gamma+1}{\gamma-1}\partial_{t_2}\pi^{(0)}_{\alpha\beta} + \frac{\gamma^2+4\gamma+1}{3(\gamma-1)^2}\partial_{t_1}\left(\pi^{*(1)}_{\alpha\beta} - \pi^{(1)}_{\alpha\beta}\right)\right] \\
&+ \varepsilon^2 \frac{h}{2}\partial_\beta \left[-h\frac{\gamma+1}{(\gamma-1)^2}\partial_{t_1}\Sigma^{(1)}_{\alpha\beta} - \frac{2}{\gamma-1}\Sigma^{(2)}_{\alpha\beta}\right] - \varepsilon^3 \frac{1}{12}h^2\partial^2_{t_1} f^{(1)}_\alpha \\
&+ \varepsilon^2 \frac{h}{2}\partial_\beta \left[\frac{\gamma^2+4\gamma+1}{3(\gamma-1)^2}\partial_{\gamma_1}\left(\phi^{*(1)}_{\alpha\beta\gamma} - \phi^{(1)}_{\alpha\beta\gamma}\right) - h\frac{\gamma+1}{(\gamma-1)^2}\partial_{\gamma_1}\Xi^{(1)}_{\alpha\beta\gamma}\right] \\
&= \varepsilon f^{(1)}_\alpha + \varepsilon^2 f^{(2)}_\alpha + \varepsilon^3 f^{(3)}_\alpha
\end{aligned}
$$

Navier-Stokes equation

$$
\begin{aligned}
&\partial_t j_\alpha + \partial_\beta \pi^{(0)}_{\alpha\beta} + \varepsilon \frac{1}{2}\frac{\gamma+1}{\gamma-1}\partial_\beta \left(\pi^{*(1)}_{\alpha\beta} - \pi^{(1)}_{\alpha\beta}\right) - \varepsilon \frac{h}{\gamma-1}\partial_\beta \Sigma^{(1)}_{\alpha\beta} \\
&+ \varepsilon^2 \frac{h}{2}\partial_\beta \left[\frac{\gamma+1}{\gamma-1}\partial_{t_2}\pi^{(0)}_{\alpha\beta} + \frac{\gamma^2+4\gamma+1}{3(\gamma-1)^2}\partial_{t_1}\left(\pi^{*(1)}_{\alpha\beta} - \pi^{(1)}_{\alpha\beta}\right)\right] \\
&+ \varepsilon^2 \frac{h}{2}\partial_\beta \left[-h\frac{\gamma+1}{(\gamma-1)^2}\partial_{t_1}\Sigma^{(1)}_{\alpha\beta} - \frac{2}{\gamma-1}\Sigma^{(2)}_{\alpha\beta}\right] - \varepsilon^3 \frac{1}{12}h^2 \partial^2_{t_1} f^{(1)}_\alpha \\
&+ \varepsilon^2 \frac{h}{2}\partial_\beta \left[\frac{\gamma^2+4\gamma+1}{3(\gamma-1)^2}\partial_{\gamma_1}\left(\phi^{*(1)}_{\alpha\beta\gamma} - \phi^{(1)}_{\alpha\beta\gamma}\right) - h\frac{\gamma+1}{(\gamma-1)^2}\partial_{\gamma_1}\Xi^{(1)}_{\alpha\beta\gamma}\right] \\
&= \varepsilon f^{(1)}_\alpha + \varepsilon^2 f^{(2)}_\alpha + \varepsilon^3 f^{(3)}_\alpha
\end{aligned}
$$

Navier-Stokes equation with force at 3rd order taking into account interfaces

Navier-Stokes equation

$$\partial_t j_\alpha + \partial_\beta \pi^{(0)}_{\alpha\beta} + \varepsilon \frac{1}{2}\frac{\gamma+1}{\gamma-1}\partial_\beta \left(\pi^{*(1)}_{\alpha\beta} - \pi^{(1)}_{\alpha\beta}\right) - \varepsilon \frac{h}{\gamma-1}\partial_\beta \Sigma^{(1)}_{\alpha\beta}$$
$$+ \varepsilon^2 \frac{h}{2}\partial_\beta \left[\frac{\gamma+1}{\gamma-1}\partial_{t_2}\pi^{(0)}_{\alpha\beta} + \frac{\gamma^2+4\gamma+1}{3(\gamma-1)^2}\partial_{t_1}\left(\pi^{*(1)}_{\alpha\beta} - \pi^{(1)}_{\alpha\beta}\right)\right]$$
$$+ \varepsilon^2 \frac{h}{2}\partial_\beta \left[-h\frac{\gamma+1}{(\gamma-1)^2}\partial_{t_1}\Sigma^{(1)}_{\alpha\beta} - \frac{2}{\gamma-1}\Sigma^{(2)}_{\alpha\beta}\right] - \varepsilon^3 \frac{1}{12}h^2\partial^2_{t_1} f^{(1)}_\alpha$$
$$+ \varepsilon^2 \frac{h}{2}\partial_\beta \left[\frac{\gamma^2+4\gamma+1}{3(\gamma-1)^2}\partial_{\gamma_1}\left(\phi^{*(1)}_{\alpha\beta\gamma} - \phi^{(1)}_{\alpha\beta\gamma}\right) - h\frac{\gamma+1}{(\gamma-1)^2}\partial_{\gamma_1}\Xi^{(1)}_{\alpha\beta\gamma}\right]$$
$$= \varepsilon f^{(1)}_\alpha + \varepsilon^2 f^{(2)}_\alpha + \varepsilon^3 f^{(3)}_\alpha$$

Adjust correction collision operator $\Rightarrow$ undesired terms $= 0$

$$h\Sigma_{\alpha\beta} = \sum_i \Delta_i^{corr} c_{i\alpha} c_{i\beta} \qquad h\Xi_{\alpha\beta\gamma} = \sum_i \Delta_i^{corr} c_{i\alpha} c_{i\beta} c_{i\gamma}$$

Navier-Stokes equation

$$\partial_t j_\alpha + \partial_\beta \pi^{(0)}_{\alpha\beta} + \varepsilon \frac{1}{2} \frac{\gamma+1}{\gamma-1} \partial_\beta \left(\pi^{*(1)}_{\alpha\beta} - \pi^{(1)}_{\alpha\beta} \right)$$

$$= \qquad \varepsilon^3 f^{(3)}_\alpha$$

Navier-Stokes equation with force at 3rd order taking into account interfaces

Summary

- Good news
  - no spurious terms
  - 6th rank isotropy
  - fully consistent method for gas-liquid coexistence can be constructed
- Bad news
  - many velocities needed (21 in 2D, 59 in 3D) + another set of velocities for interface collision operator (12 in 2D, 20 in 3D)
  - need for finite difference scheme to compute the correction collision operator
  - limited freedom for choosing the equation of state
  - need of iterative algorithm to calculate flow velocity
- Future
  - test numerical implementation
  - thermal fluctuations
  - boundaries
  - binary fluids