# On the effective string theory of confining flux tubes

Michael Teper (Oxford) - KITP, 2012

- Flux tubes and string theory :

  effective string theories - recent progress

  fundamental flux tubes in D=2+1

  fundamental flux tubes in D=3+1

  higher representation flux tubes

- Concluding remarks

## gauge theory and string theory

$\leftrightarrow$

A long history ...

- Veneziano amplitude
- 't Hooft large-$N$ – genus diagram expansion
- Polyakov action
- Maldacena ... AdS/CFT/QCD ...

at large $N$, flux tubes and perhaps the whole gauge theory can be described by a weakly-coupled string theory

we calculate the spectrum of closed flux tubes:
— closed around a spatial torus of length $l$ —

- flux localised in 'tubes'; long flux tubes, $l\sqrt{\sigma} \gg 1$ look like 'thin strings'
- at $l = l_c = 1/T_c$ there is a 'deconfining' phase transition: 1st order for $N \geq 3$ in $D = 4$ and for $N \geq 4$ in $D = 3$
- so may have a simple string description of the closed string spectrum for all $l \geq l_c$
- most plausible at $N \to \infty$ where scattering, mixing and decay, e.g string $\to$ string + glueball, go away
- in both D=2+1 and D=3+1

Note: the static potential $V(r)$ describes the transition in $r$ between UV (Coulomb potential) and IF (flux tubes) physics; potentially of great interest as $N \to \infty$.

analytic work:

Luscher and Weisz, hep-th/0406205; Drummond, hep-th/0411017.

Aharony with Karzbrun, Field, Klinghoffer, Dodelson, arXiv:0903.1927; 1008.2636; 1008.2648; 1111.5757; 1111.5758

numerical work:

closed flux tubes:
Athenodorou, Bringoltz, MT, arXiv:1103.5854, 1007.4720, ... ,0802.1490, 0709.0693

Wilson loops and open flux tubes:
Caselle, Gliozzi, et al ..., arXiv:1202.1984, 1107.4356, ...

also
Brandt, arXiv:1010.3625; Lucini,..., 1101.5344; ......

historical aside:

QCD and String Theory, KITP 2004

Nair's analytic prediction in D=2+1:

$$\frac{\sqrt{\sigma}}{g^2 N} = \sqrt{\frac{1-1/N^2}{8\pi}} \stackrel{N\to\infty}{\to} 0.19947 - \frac{0.0998}{N^2}$$

versus my 1998 lattice calculation:

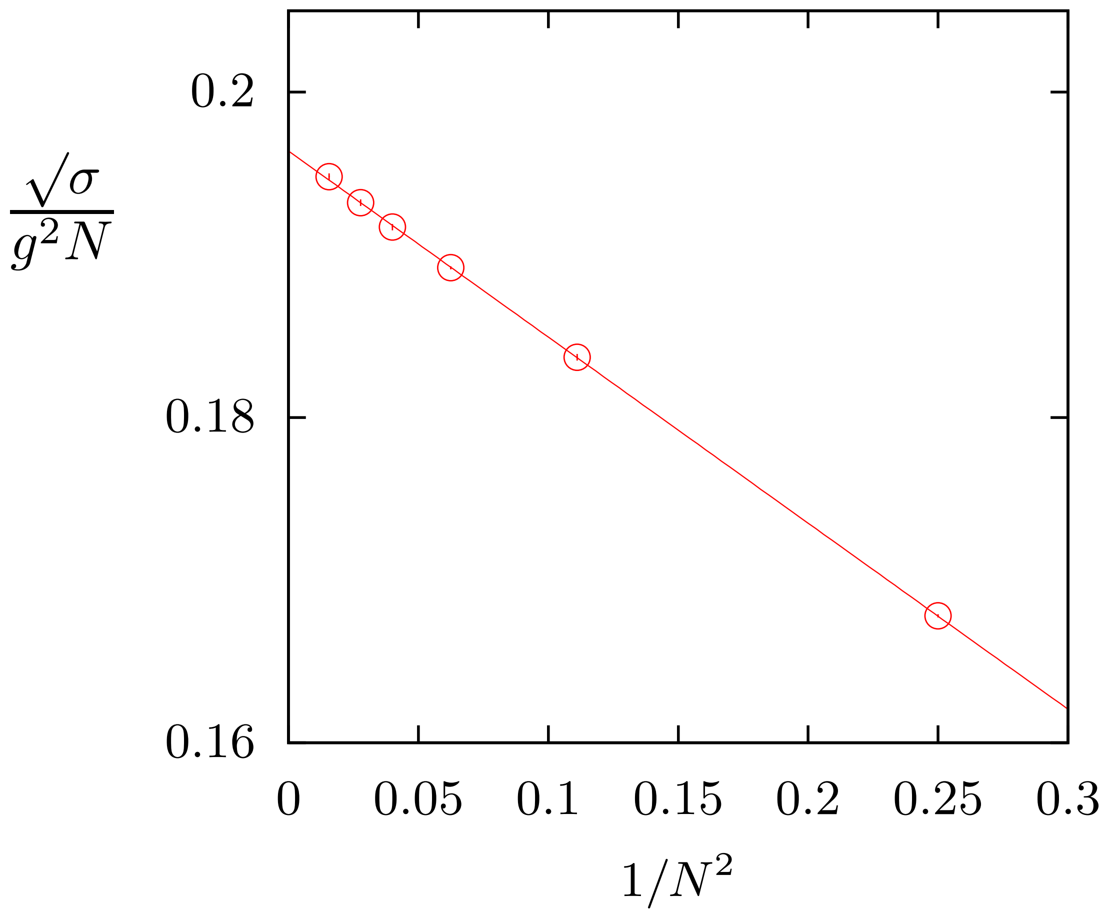
$$\frac{\sqrt{\sigma}}{g^2 N} \stackrel{N\to\infty}{\to} 0.1975(10) - \frac{0.119(8)}{N^2}$$

perhaps they actually agree?

$\Longrightarrow$

need better control systematic errors, in particular the $l$-dependence of the flux tube energy ....

continuum limits of $N \in [2,8]$ in $D = 2+1$ Bringoltz,MT hep-th/0611286




fit: $\lim_{N\to\infty} \frac{\sqrt{\sigma}}{g^2 N} = 0.1975(\pm 2)(-5)$ i.e. $\sim 1\%$ $\sim 8\sigma$ less than Nair, Athenodorou,Bringoltz,MT ArXiv:1103.5854

'test' large $N$ counting

$\Longrightarrow$

$$\frac{\sqrt{\sigma}}{g^2 N} = c_0 + \frac{c_1}{N^\gamma} \quad \Rightarrow \quad \gamma = 1.97 \pm 0.10$$

$$\frac{\sqrt{\sigma}}{g^2 N^\alpha} = c_0 + \frac{c_1}{N^2} \quad \Rightarrow \quad \alpha = 1.002 \pm 0.004$$

$$\frac{\sqrt{\sigma}}{g^2 N^\alpha} = c_0 + \frac{c_1}{N^\gamma} \quad \Rightarrow \quad \alpha = 1.008 \pm 0.015, \; \gamma = 2.18 \pm 0.40$$

$\Longrightarrow$

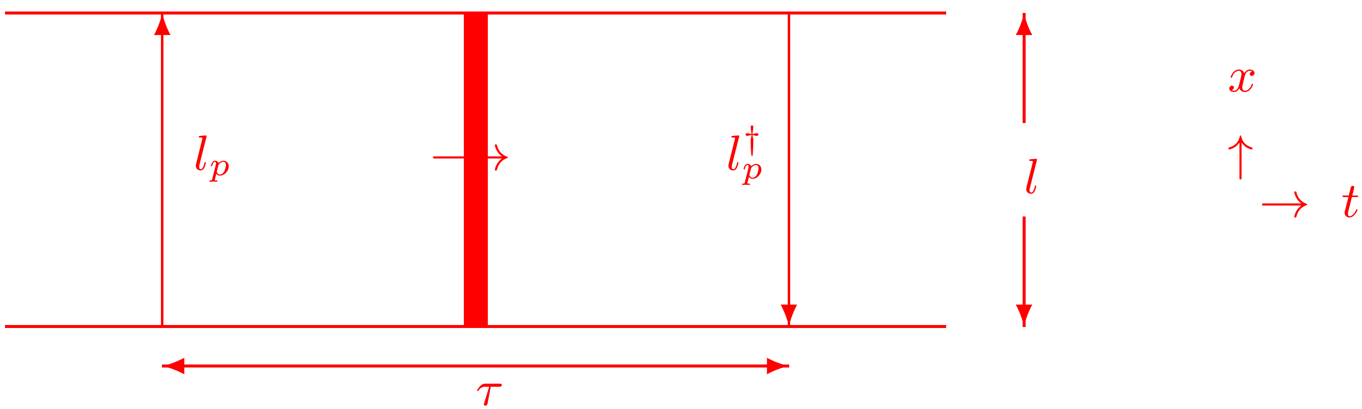
strong support for non-perturbative validity of usual large-$N$ counting i.e.

$$\frac{\sqrt{\sigma}}{g^2 N} = c_0 + \frac{c_1}{N^2} + \cdots$$

calculate the energy spectrum of a confining flux tube winding around a spatial torus of length $l$, using correlators of Polyakov loops (Wilson lines):

$$\langle l_p^\dagger(\tau) l_p(0)\rangle = \sum_{n,p_\perp} c_n(p_\perp, l) e^{-E_n(p_\perp,l)\tau} \overset{\tau\to\infty}{\propto} \exp\{-E_0(l)\tau\}$$

in pictures

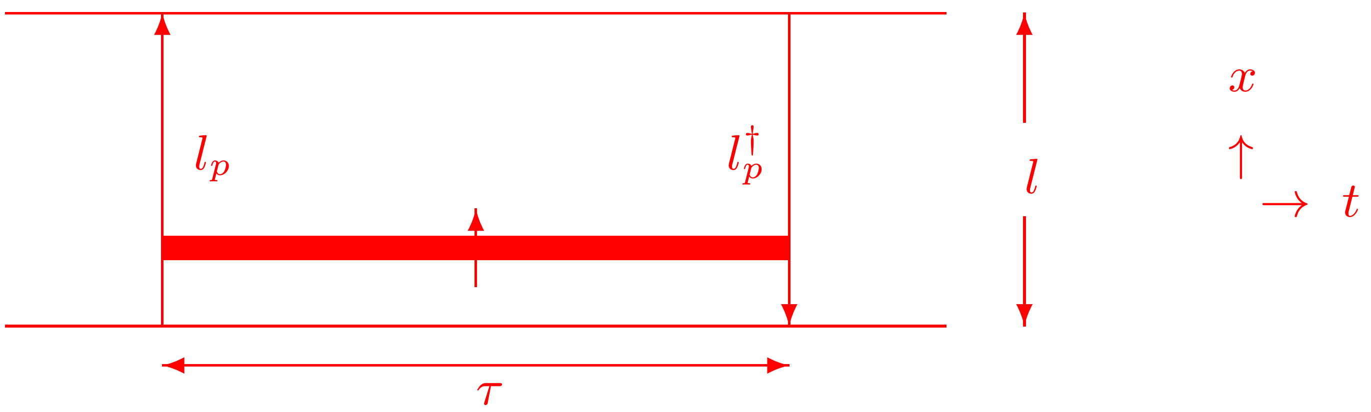



a flux tube sweeps out a cylindrical $l \times \tau$ surface $S$ $\cdots$ integrate over these world sheets with an effective string action $\propto \int_{cyl=l\times\tau} dS e^{-S_{eff}[S]}$

also a flux tube attached to the static sources propagating in the $x$-direction:

$$\langle l_p^\dagger(\tau) l_p(0)\rangle = \sum_n e^{-\hat{E}_n(\tau) l} \overset{l\to\infty}{\propto} \exp\{-\hat{E}_0(\tau) l\}$$

in pictures




this is an example of an 'open-closed string duality'

$\Rightarrow$

$$\langle l_p^\dagger(\tau) l_p(0)\rangle = \sum_{n,p_\perp} c_n(p_\perp, l) e^{-E_n(p_\perp,l)\tau} = \sum_n e^{-\hat{E}_n(\tau) l} = \int_{cyl=l\times\tau} dS e^{-S_{eff}[S]}$$

where $S_{eff}[S]$ is the effective string action for the surface $S$

$\Rightarrow$

the string partition function will predict the spectrum $\hat{E}_n(\tau)$ – just a Laplace transform – but will be constrained by the Lorentz invariance encoded in $E_n(p_\perp, l)$

Luscher and Weisz; Meyer

this can be extended from a cylinder to a torus (Aharony)

$$Z^{w=1}_{torus}(l,\tau) = \sum_{n,p} e^{-E_n(p,l)\tau} = \sum_{n,p} e^{-E_n(p,\tau)l} = \int_{T^2=l\times\tau} dSe^{-S_{eff}[S]}$$

where $p$ now includes both transverse and longitudinal momenta

$\leftrightarrow$

'closed-closed string duality'

Parameterising $S$ (static gauge):

• $h(x,t)$ is transverse displacement (vector in $D = 3 + 1$) from minimal surface $x \in [0,l]$ and $t \in [0,\tau]$, i.e.

$$S_{eff}[S] \longrightarrow S_{eff}[h]$$

and we integrate over the field $h(x,t)$

• translation invariance $\Rightarrow$ $S_{eff}[h]$ cannot depend on position but only on $\partial_\alpha h$, with $\alpha = x, t$, $\Rightarrow$ we can do a derivative expansion (schematic):

$$S_{eff} \sim \sigma l \tau + \int_0^\tau dt \int_0^l dx \tfrac{1}{2} \partial h \partial h + \sum c_{n,i} \int_0^\tau dt \int_0^l dx \partial^{n+i} h^n$$

$\Rightarrow$ an expansion of $E_n(l)$ in powers of $1/\sigma l^2$

• open-closed duality constrains some of these coefficients $\Rightarrow$ some correction terms in $E(l) = \sigma l + \frac{c_1}{l} + \frac{c_2}{\sigma l^3} + \cdots$ are 'universal'

e.g. $c_1 = \pi(D-2)/6$ – the famous Luscher correction

So what do we know?

any $S_{eff}$ $\Rightarrow$

$$E_0(l) \stackrel{l\to\infty}{=} \sigma l - \frac{\pi(D-2)}{6l} - \frac{\{\pi(D-2)\}^2}{72}\frac{1}{\sigma l^3} - \frac{\{\pi(D-2)\}^3}{432}\frac{1}{\sigma^2 l^5} + O\left(\frac{1}{l^7}\right)$$

universal terms:

- $O\left(\frac{1}{l}\right)$ Luscher correction, $\sim$ 1980
- $O\left(\frac{1}{l^3}\right)$ Luscher, Weisz; Drummond, $\sim$ 2004
- $O\left(\frac{1}{l^5}\right)$ Aharony et al, $\sim$ 2009-10

and similar results for $E_n(l)$, but only to $O(1/l^3)$ in $D = 3+1$

just like the simple free string theory
: Nambu-Goto in flat space-time up to explicit $O(1/l^7)$ corrections

So what does one find numerically?

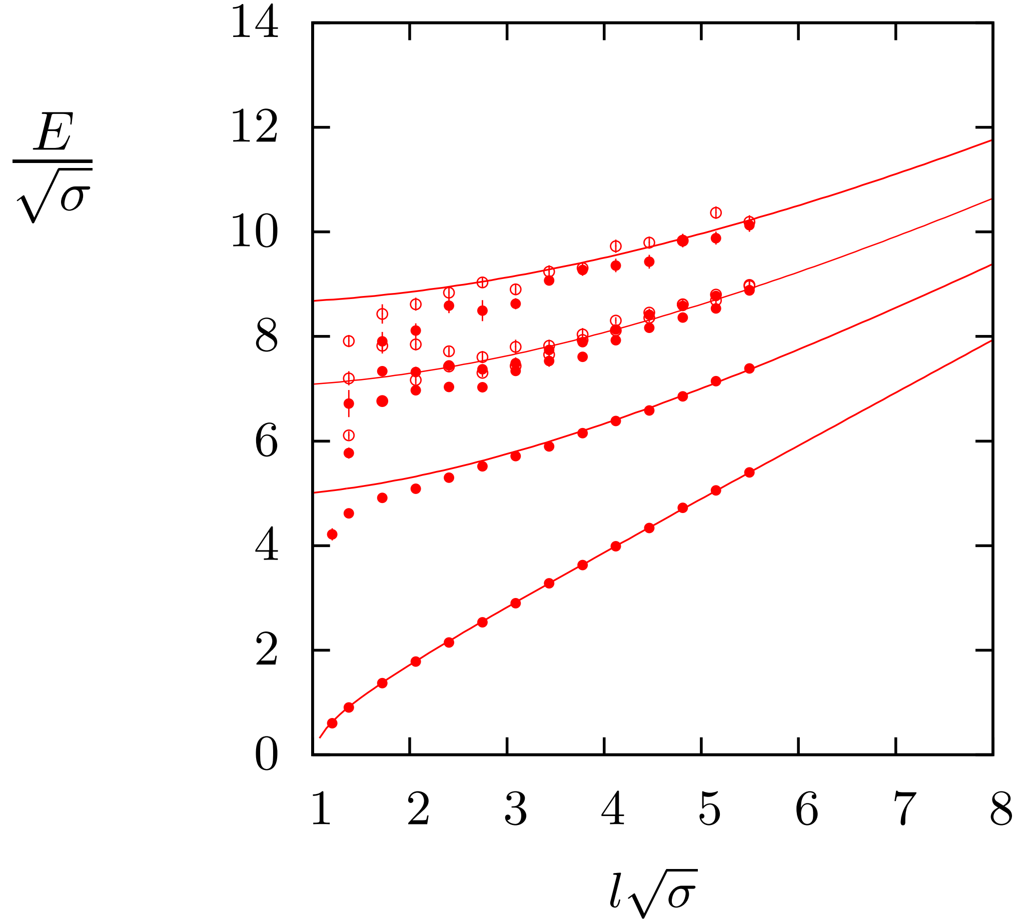
results here are from:

- $D = 2 + 1$ Athenodorou, Bringoltz, MT, arXiv:1103.5854
- $D = 3 + 1$ Athenodorou, Bringoltz, MT, arXiv:1007.4720
- higher rep Athenodorou, MT, in progress

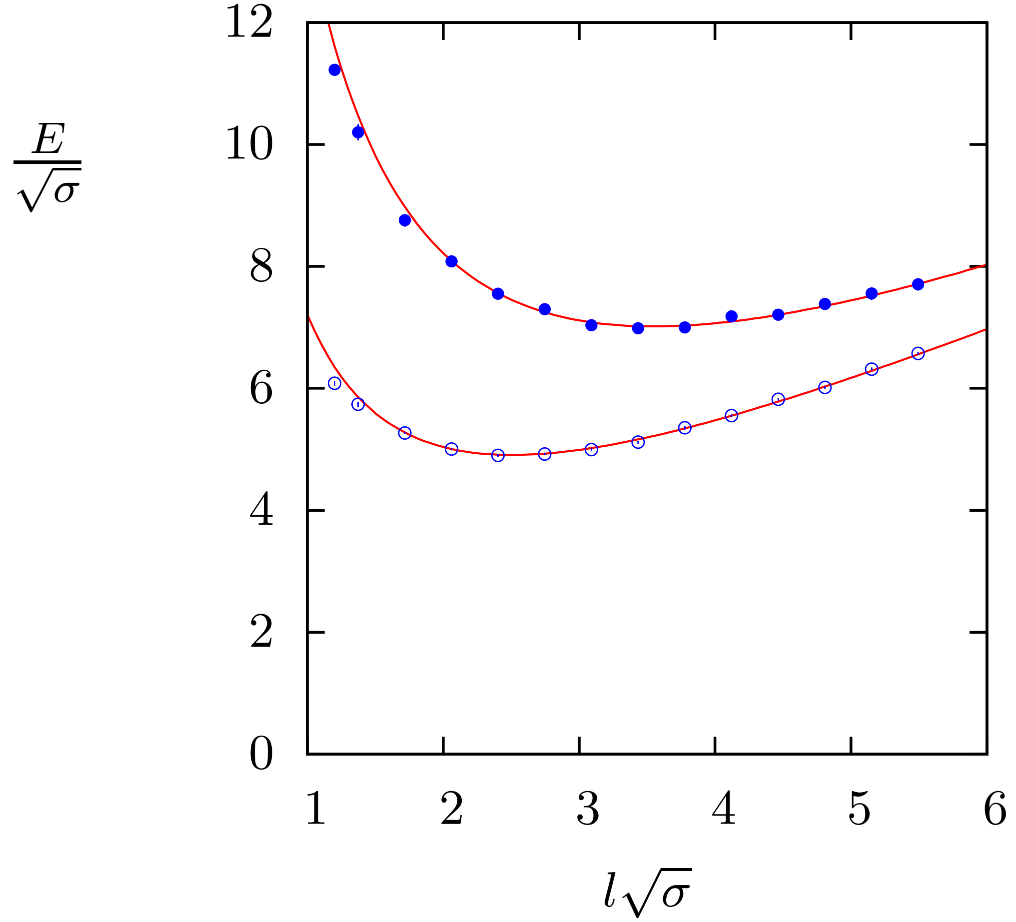
and we start with:

$D = 2 + 1,\ SU(6),\ a\sqrt{\sigma} \simeq 0.086$ i.e $N \sim \infty,\ a \sim 0$

lightest 8 states with $p = 0$ $P = +(\bullet), P = -(\circ)$




solid lines: Nambu-Goto ground state $\to$ $\sigma$: only parameter lightest levels with $p = 2\pi q/l, \; 4\pi q/l$ $P = -$




Nambu-Goto : solid lines

## Nambu-Goto free string theory

$$\int \mathcal{D}S e^{-\kappa A[S]}$$

spectrum (Arvis 1983, Luscher-Weisz 2004):

$$E^2(l) = (\sigma\, l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{D-2}{24} \right) + \left( \frac{2\pi q}{l} \right)^2 .$$

$p = 2\pi q/l =$ total momentum along string;

$N_L, N_R =$ sum left and right 'phonon' momentum:

$$N_L = \sum_{k>0} n_L(k)\, k, \quad N_R = \sum_{k>0} n_R(k)\, k, \quad N_L - N_R = q$$

where

$$\text{state} = \prod_{k>0} a_k^{n_L(k)} a_{-k}^{n_R(k)} |0\rangle \quad , \quad P = (-1)^{number\ phonons}$$

lightest $p = 0$ states:

$|0\rangle$

$a_1 a_{-1}|0\rangle$

$a_2 a_{-2}|0\rangle,\ a_2 a_{-1} a_{-1}|0\rangle,\ a_1 a_1 a_{-2}|0\rangle,\ a_1 a_1 a_{-1} a_{-1}|0\rangle$

$\ldots$

lightest $p \neq 0$ states:

| | |
|---|---|
| $a_1|0\rangle$ | $P = -,\ p = 2\pi/l$ |
| $a_2|0\rangle$ | $P = -,\ p = 4\pi/l$ |
| $a_1 a_1|0\rangle$ | $P = +,\ p = 4\pi/l$ |

$\Rightarrow$

observe Nambu-Goto degeneracies and quantum numbers

Since when Nambu-Goto is expanded the first few terms are universal e.g.

$$
\begin{aligned}
E_0(l) &= \sigma l \left(1 - \frac{\pi(D-2)}{3\sigma l^2}\right)^{\frac{1}{2}} \\
&\stackrel{l > l_0}{=} \sigma l - \frac{\pi(D-2)}{6l} - \frac{\{\pi(D-2)\}^2}{72}\frac{1}{\sigma l^3} - \frac{\{\pi(D-2)\}^3}{432}\frac{1}{\sigma^2 l^5} + O\left(\frac{1}{l^7}\right)
\end{aligned}
$$

and also for excited states, e.g.

$$
E_n(l) = \sigma l \left(1 + \frac{8\pi}{\sigma l^2}\left(n - \frac{D-2}{24}\right)\right)^{\frac{1}{2}} \stackrel{l > l_n}{=} \sigma l + \sum_{n=0} \frac{c_n}{\sigma^n l^{2n+1}}
$$

where $l_0\sqrt{\sigma} = \sqrt{3/\pi(D-2)}$ and $l_n\sqrt{\sigma} \sim \sqrt{8\pi n}$

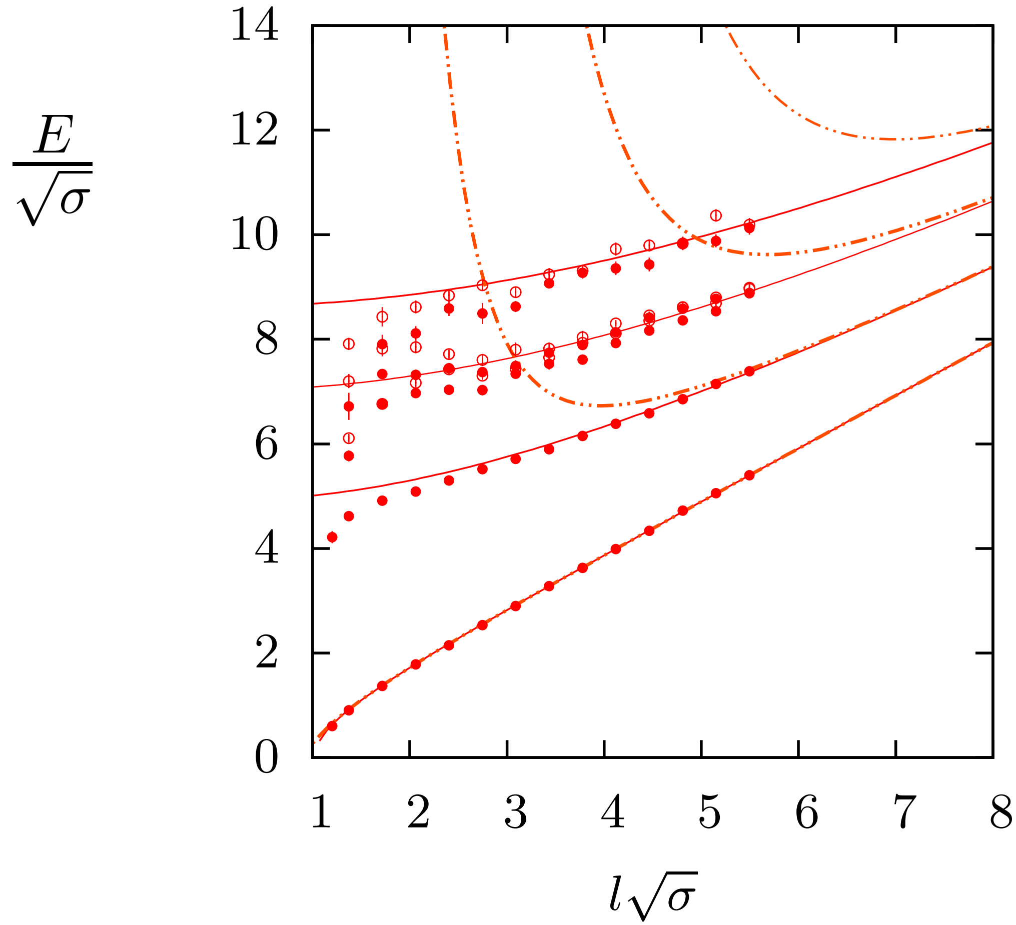
$\Rightarrow$

is the agreement with Nambu-Goto no more than agreement with the sum of the known universal terms?

NO!

universal terms: dashed lines

Nambu-Goto : solid lines



$\Longrightarrow$

• NG very good down to $l\sqrt{\sigma} \sim 2$, i.e energy

fat short flux 'tube' $\sim$ ideal thin string

• NG very good far below value of $l\sqrt{\sigma}$ where the power series expansion diverges, i.e. where all orders are important $\Rightarrow$

universal terms not enough to explain this agreeement ...

• no sign of any non-stringy modes, e.g.

$E(l) \simeq E_0(l) + \mu$ where e.g. $\mu \sim M_G/2 \sim 2\sqrt{\sigma}$

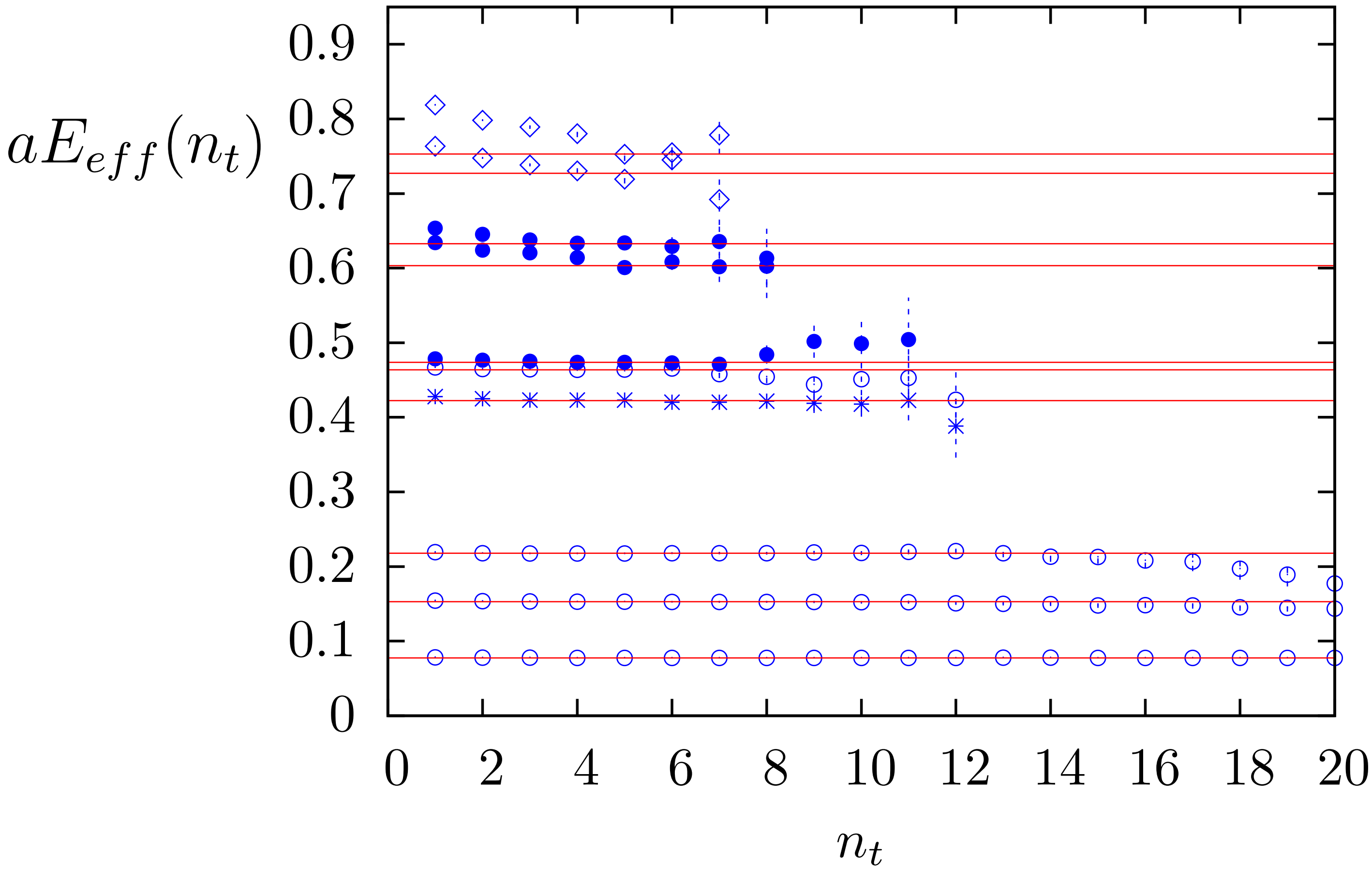
$\Longrightarrow$

... in more detail ...

but first an ‘algorithmic’ aside – calculating energies

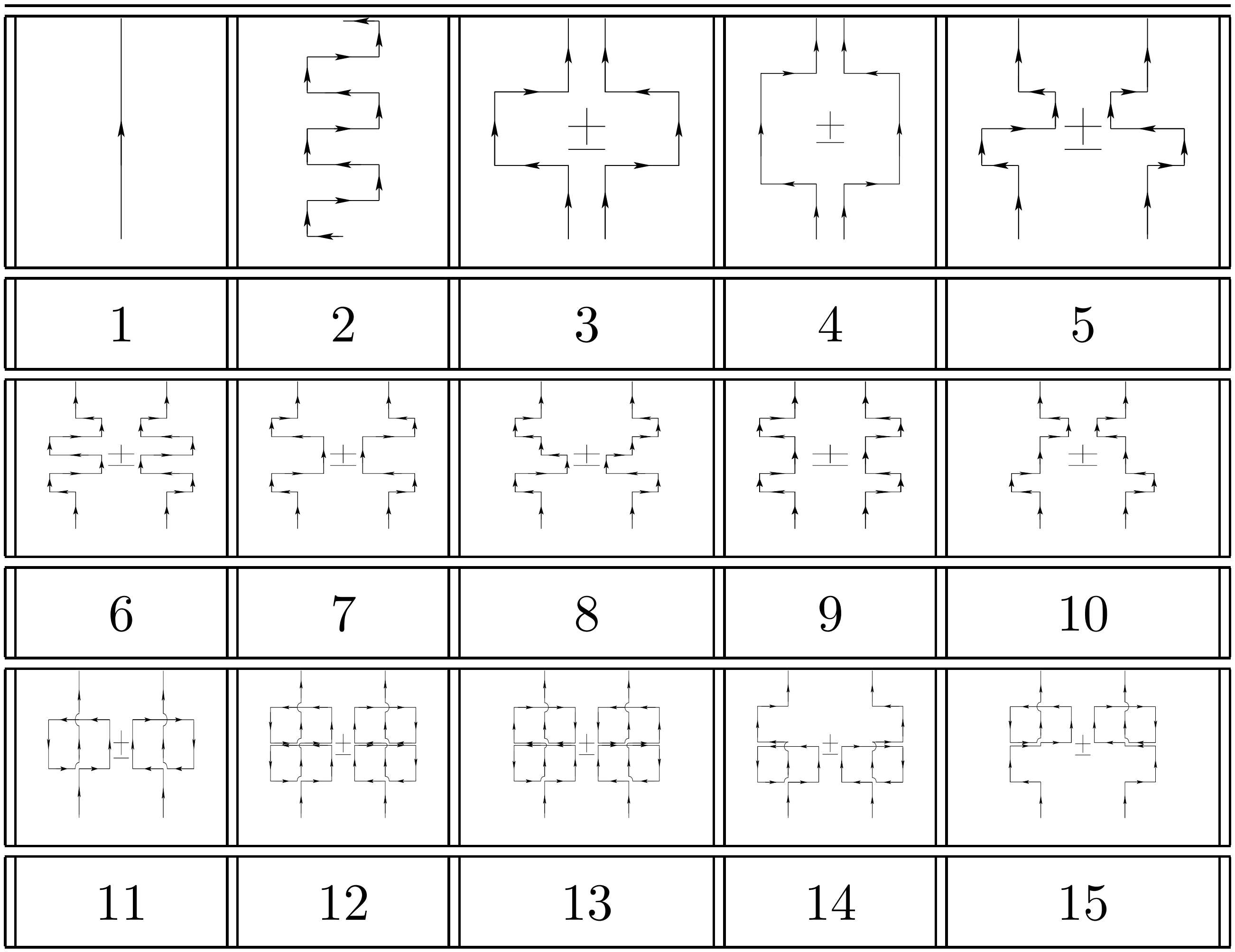
- deform Polyakov loops to allow non-trivial quantum numbers
- block or smear links to improve projection on physical excitations
- variational calculation of best operator for each energy eigenstate
- huge basis of loops for good overlap on a large number of states
- i.e. $C(t) \simeq c_n e^{-E_n(l)t}$ already for small $t$

for example:

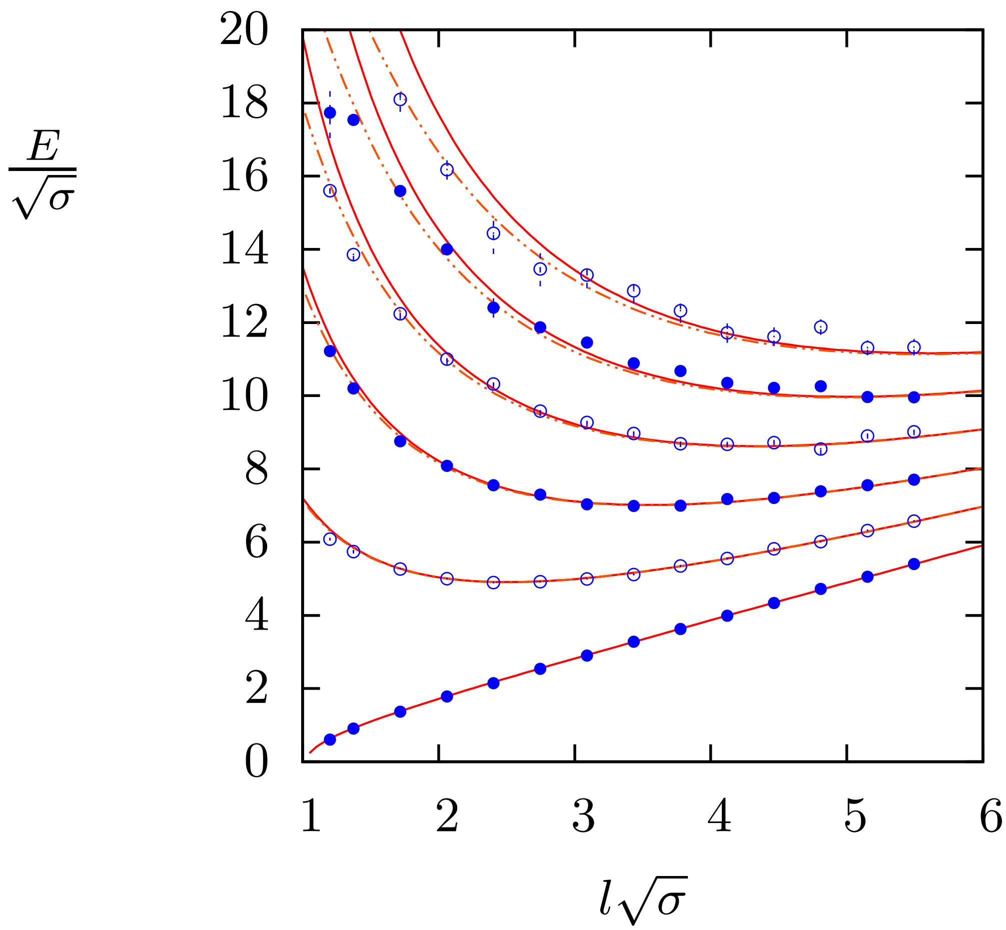



abs gs $l = 16, 24, 32, 64a$ ($\circ$); es p=0 P=+ ($\bullet$); gs $p = 2\pi/l$, $P = -$ ($\star$); gs, es $p = 0$, $P = -$ ($\diamond$)

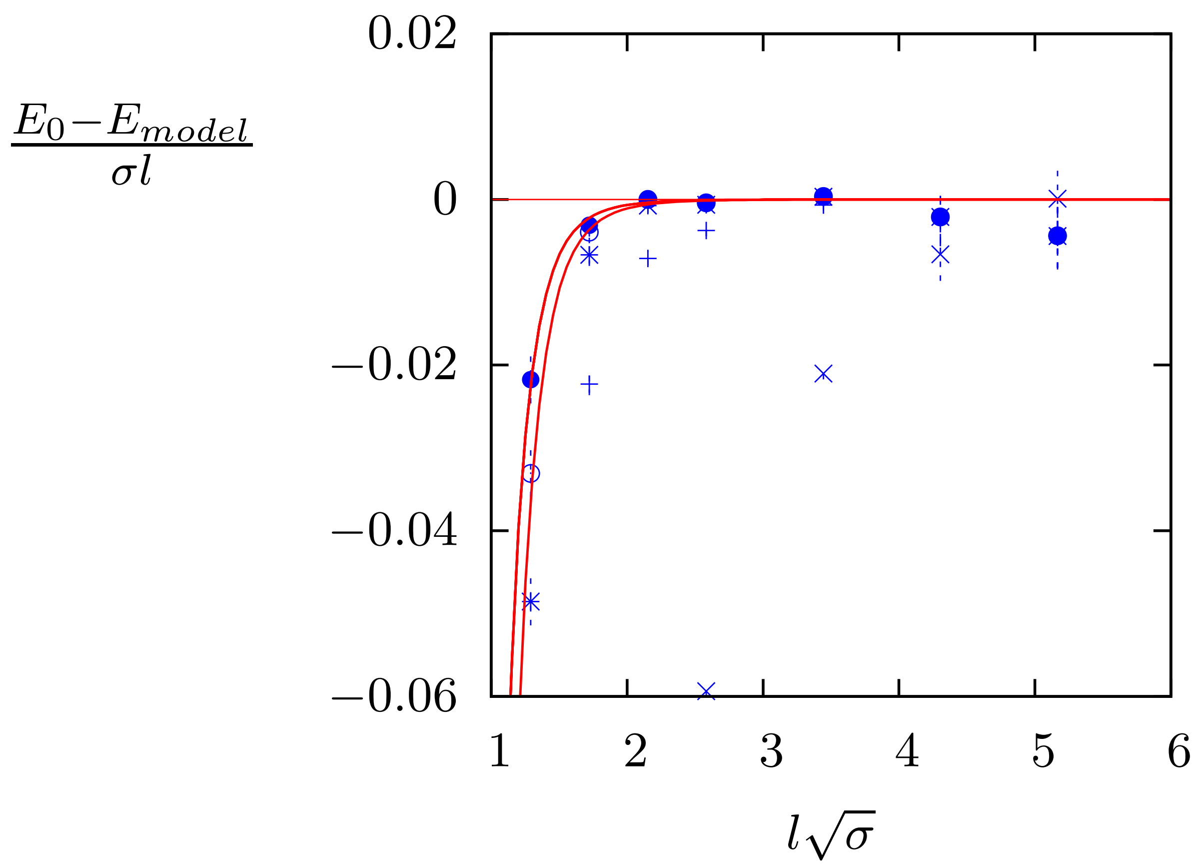
Operators in D=2+1:



lightest $P = -$ states with $p = 2\pi q/l$: $q = 0, 1, 2, 3, 4, 5$ $\qquad$ $a_q|0\rangle$




Nambu-Goto : solid lines $\qquad$ $(ap)^2 \to 2 - 2\cos(ap)$ : dashed lines ground state deviation from various 'models' $D = 2 + 1$




model = Nambu-Goto, $\bullet$, universal to $1/l^5$, $\circ$, to $1/l^3$, $\star$, to $1/l$, $+$, just $\sigma l$, $\times$

lines = plus $O(1/l^7)$ correction

$\Longrightarrow$

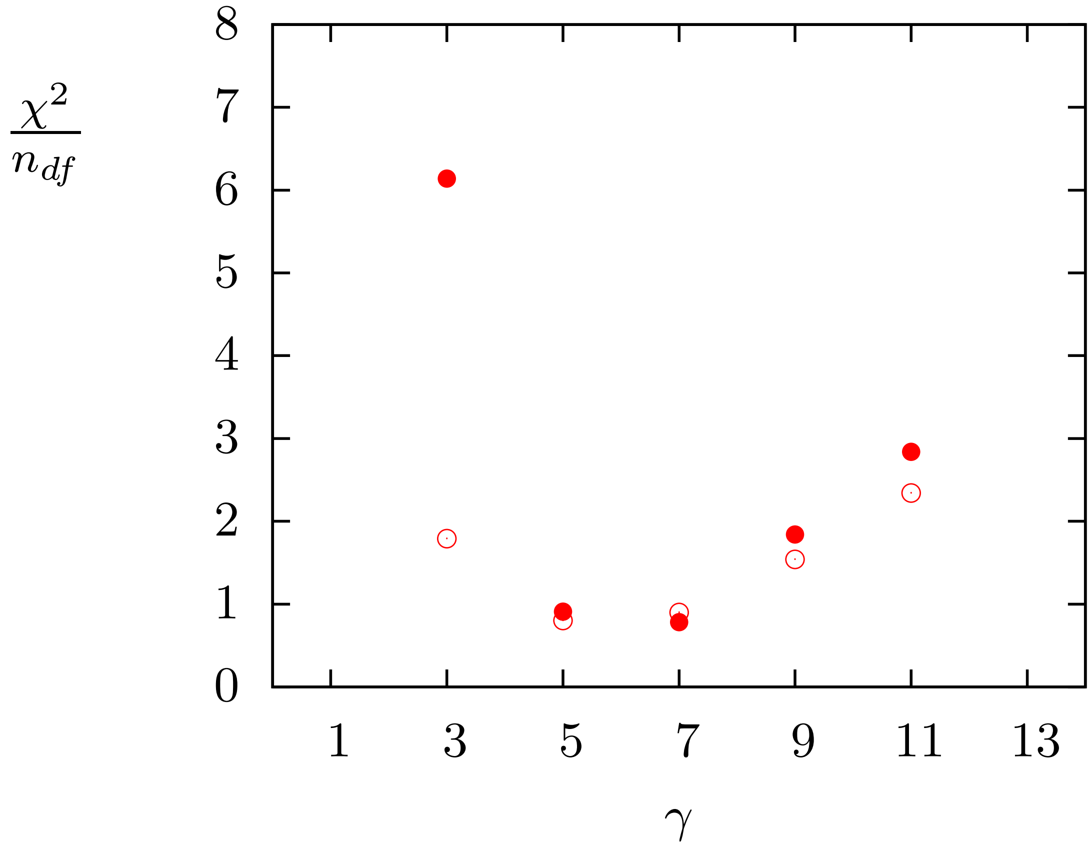
∘ for $l\sqrt{\sigma} \gtrsim 2$ agreement with NG to $\lesssim 1/1000$

moreover

∘ for $l\sqrt{\sigma} \sim 2$ contribution of NG to deviation from $\sigma l$ is $\gtrsim 99\%$

despite flux tube being short and fat

∘ and leading correction to NG consistent with $\propto 1/l^7$ as expected from current universality results




$\chi^2$ per degree of freedom for the best fit

$$E_0(l) = E_0^{NG}(l) + \frac{c}{l^\gamma}$$

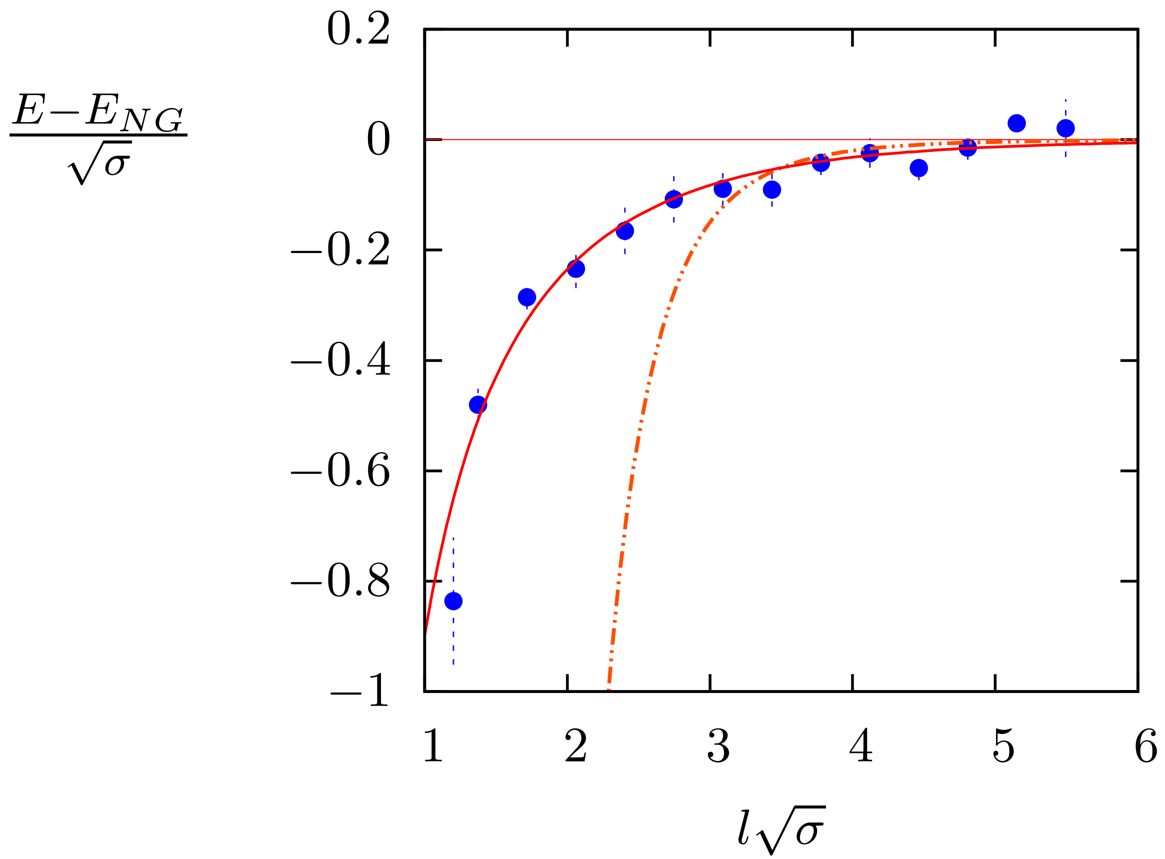
operators in expansion of $S_{NG}[h]$ are universal to all orders (Aharony: ECT talk, 2010) and so can be resummed at smaller $l$ to square root

$\Rightarrow$

we assume same is true of the corrections to NG which begin with a leading $O(1/l^7)$ term and resums at smaller $l$, i.e

$$\frac{E(l)}{\sqrt{\sigma}} = \frac{E_{NG}(l)}{\sqrt{\sigma}} + \frac{c}{(l\sqrt{\sigma})^7}\left(1 + \frac{c'}{l^2\sigma}\right)^{\gamma}$$

first excited $q = 0$, $P = +$ state $D = 2 + 1$




fits:

$\frac{c}{(l\sqrt{\sigma})^7}$ - dotted curve; $\frac{c}{(l\sqrt{\sigma})^7}\left(1 + \frac{25.0}{l^2\sigma}\right)^{-2.75}$ - solid curve

$\Longrightarrow$ if we write

$$\frac{1}{\sqrt{\sigma}} E_n(l) = \frac{1}{\sqrt{\sigma}} E_n^{NG}(l) + \frac{1}{\sqrt{\sigma}} \Delta E_n(l) \tag{1}$$

$$\stackrel{l\to\infty}{=} \frac{1}{\sqrt{\sigma}} E_n^{NG}(l) + \frac{c}{(l\sqrt{\sigma})^7} \left\{ 1 + \frac{c_1}{l^2\sigma} + \frac{c_2}{(l^2\sigma)^2} + \cdots \right\}$$

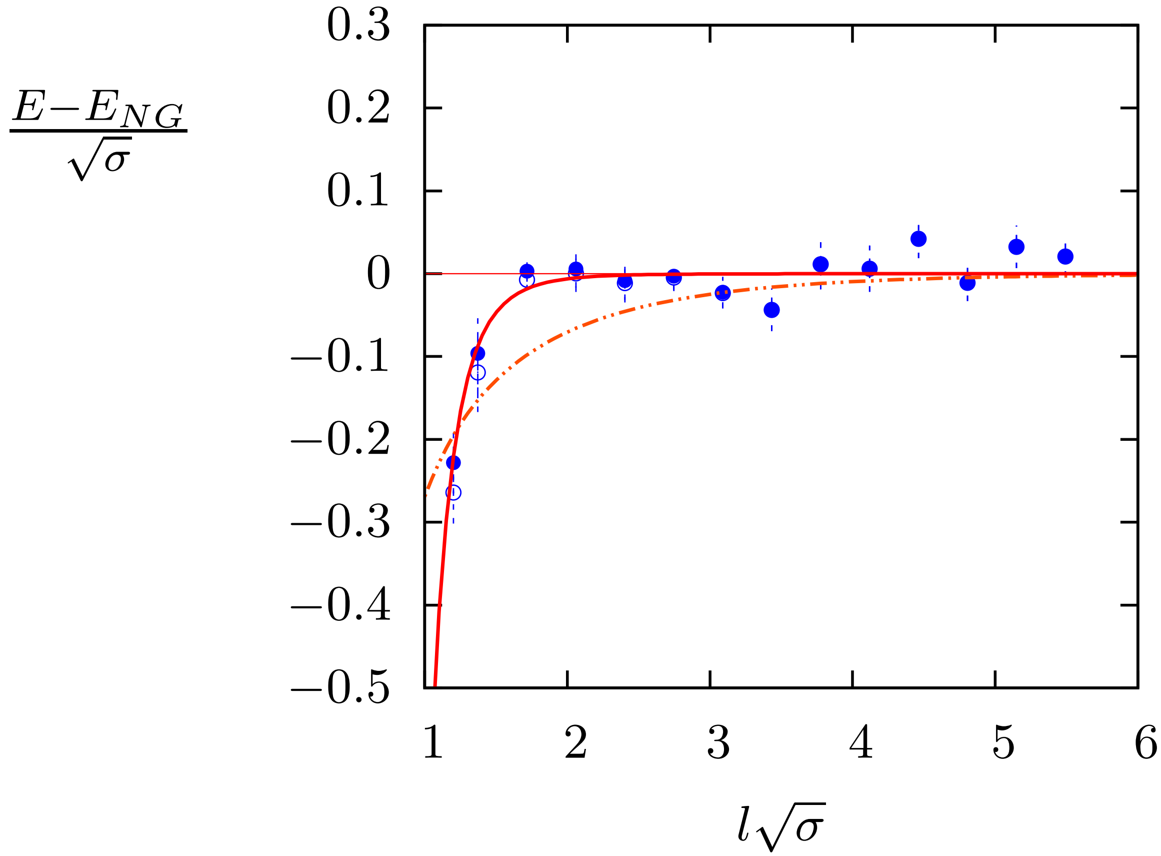
then correction to NG resums, just like NG,

$$\frac{1}{\sqrt{\sigma}} \Delta E_n(l) = \frac{c}{(l\sqrt{\sigma})^7} \left( 1 + \frac{c'}{l^2\sigma} \right)^{-\gamma} \simeq \begin{cases} \frac{c}{(l\sqrt{\sigma})^7} & l \gg l_d \\ \frac{cc'^{-\gamma}}{(l\sqrt{\sigma})^{7-2\gamma}} & l \ll l_d \end{cases}$$

and with our fit we find $c \sim 0.6 \times c_7^{NG}$

for most but not all light excited states:

$q = 1, P = -$ ground state $SU(6),\ D = 2+1$




fits:

$\frac{c}{(l\sqrt{\sigma})^7}$ solid curve; $\frac{c}{(l\sqrt{\sigma})^7}\left(1+\frac{25.0}{l^2\sigma}\right)^{-2.75}$ : dashed curve

$$D = 2 + 1 \qquad \longrightarrow \qquad D = 3 + 1$$

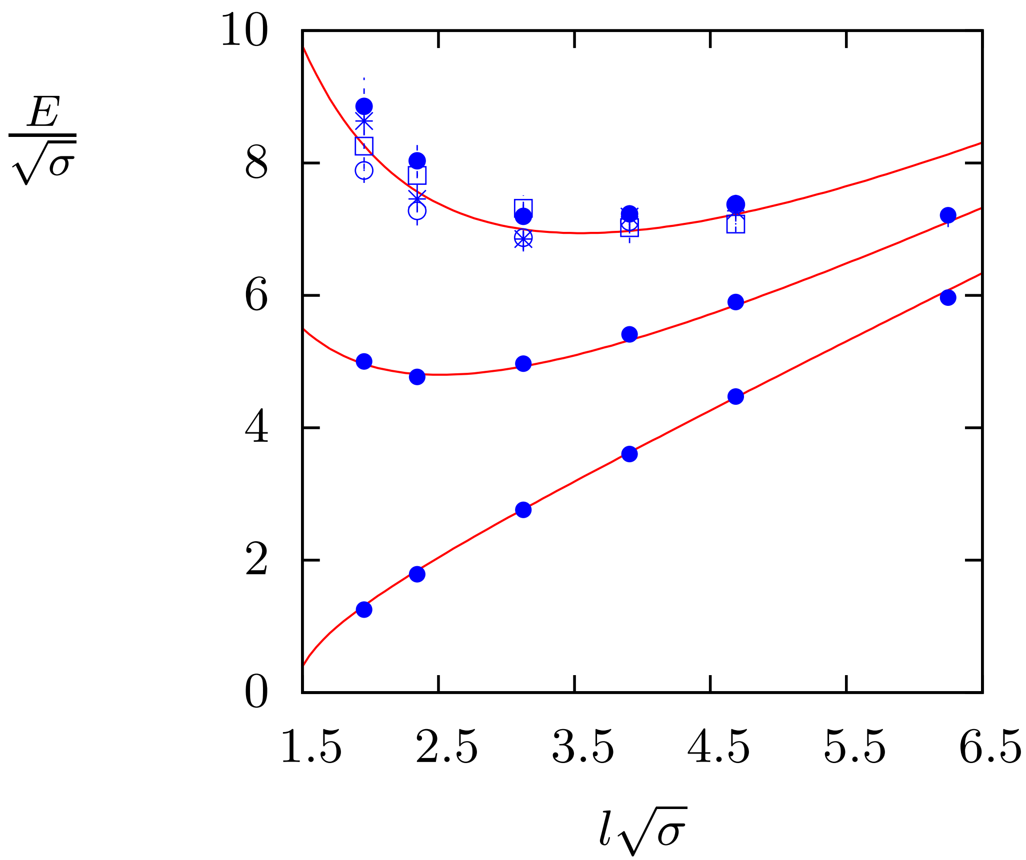
- additional rotational quantum number: phonon carries spin 1

- Nambu-Goto again remarkably good for most states

- BUT now there are some candidates for non-stringy (massive?) mode excitations ...

however in general results are considerably less accurate

$p = 2\pi q/l$ for $q = 0, 1, 2$ $\qquad$ $D = 3 + 1,\ SU(3),\ l_c\sqrt{\sigma} \sim 1.5$

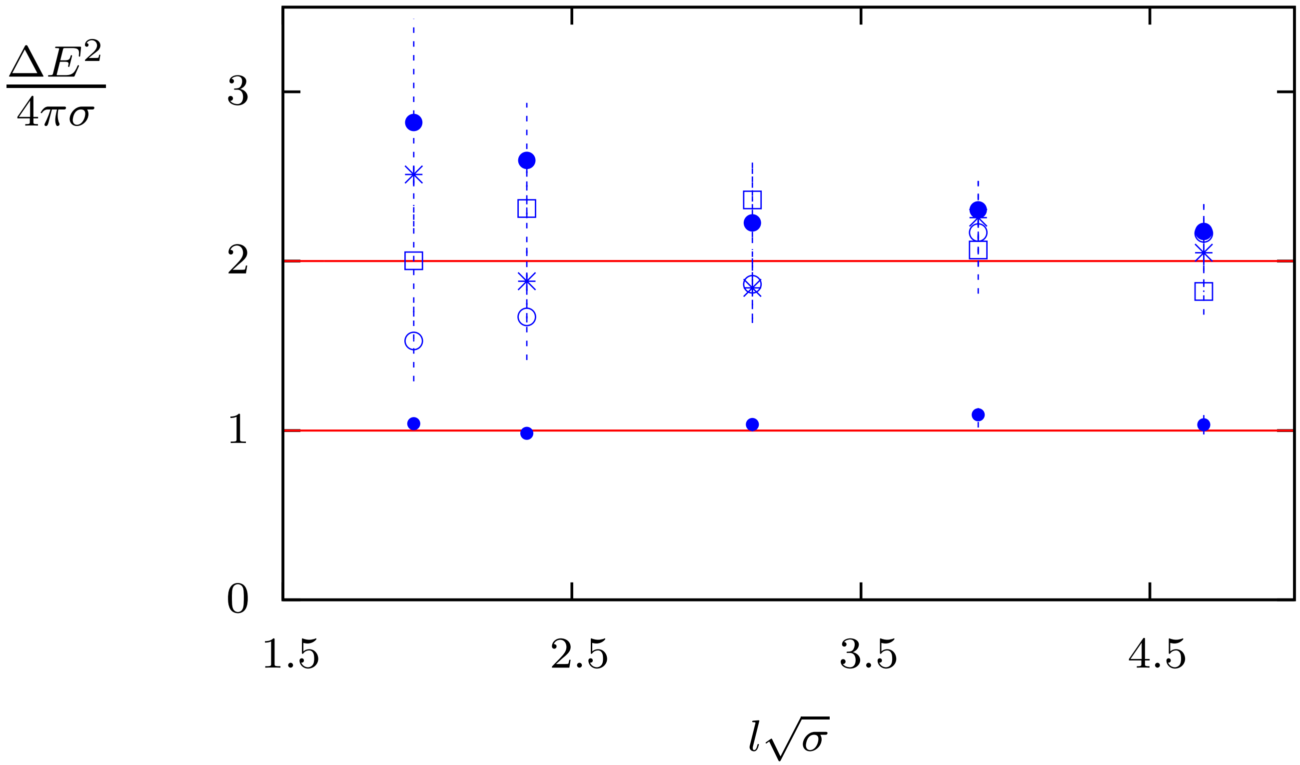



The four $q = 2$ states are: $J^{P_t} = 0^+(\star),\ 1^{\pm}(\circ),\ 2^+(\Box),\ 2^-(\bullet)$.
Lines are Nambu-Goto predictions.

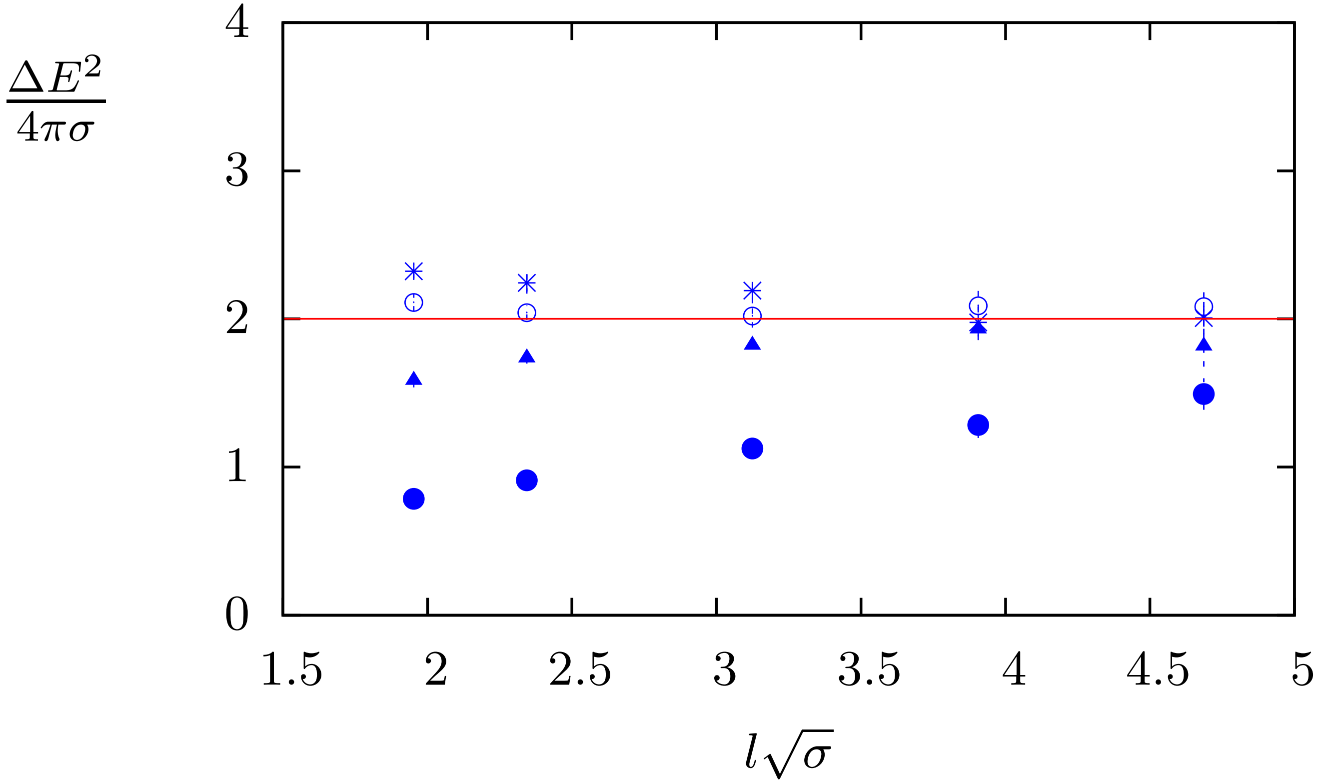
for a precise comparison with Nambu-Goto, define:

$$\Delta E^2(q,l) = E^2(q;l) - E_0^2(l) - \left(\frac{2\pi q}{l}\right)^2 \stackrel{NG}{=} 4\pi\sigma(N_L + N_R)$$

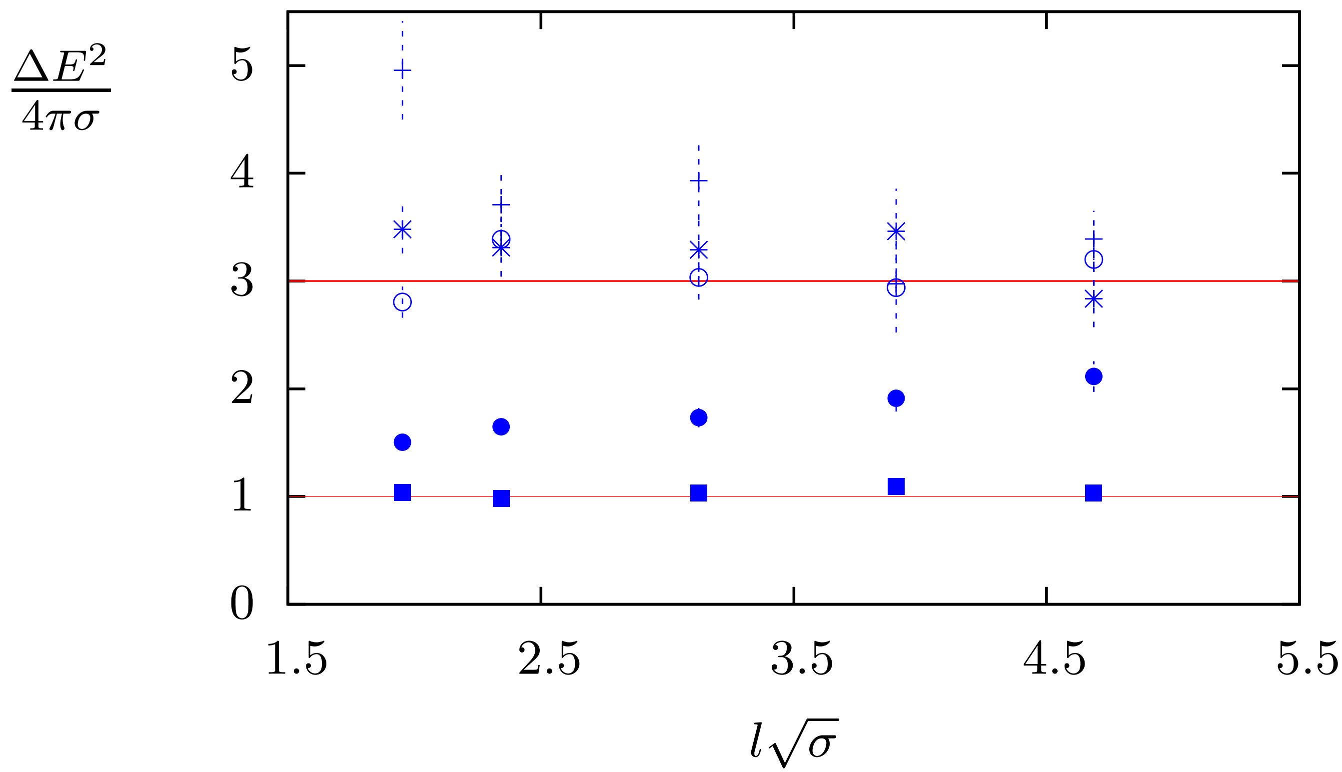
$\Longrightarrow$ lightest $q = 1, 2$ states:



lightest few $p = 0$ states



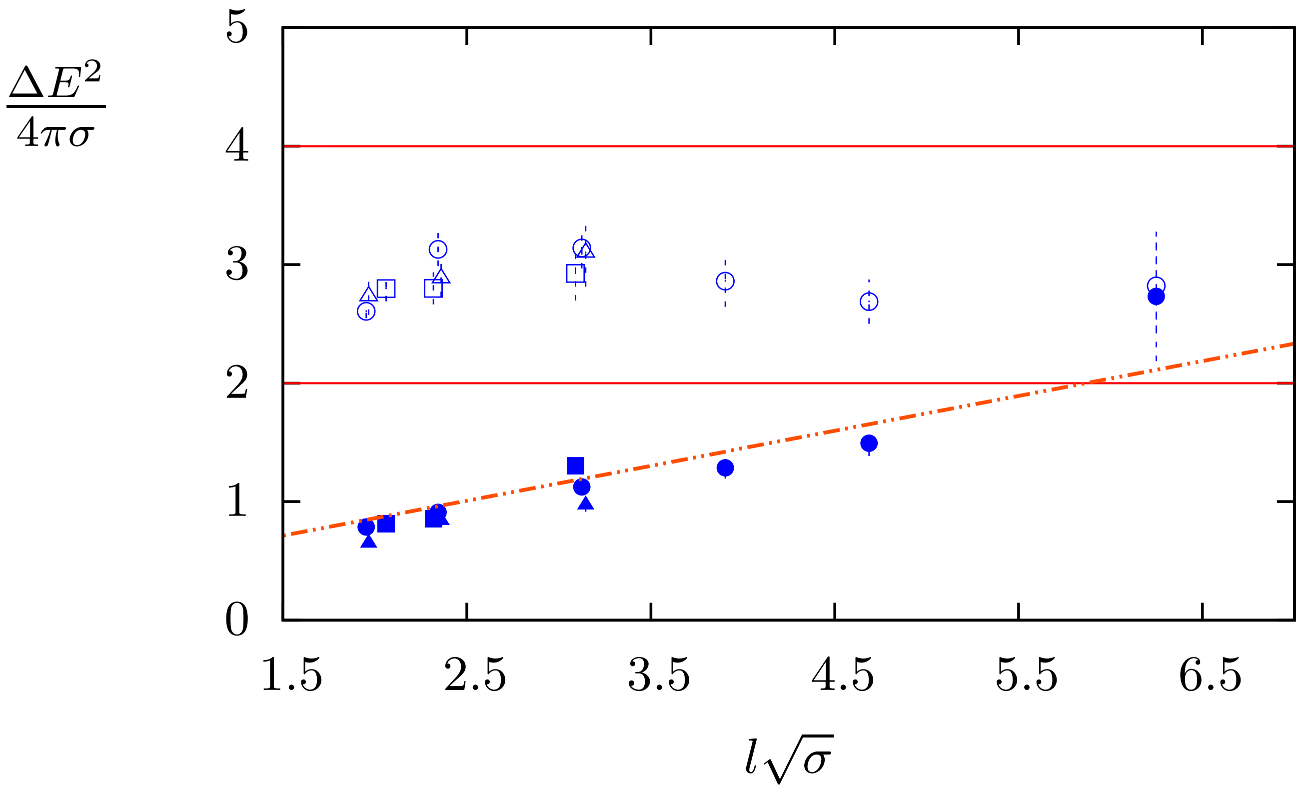

$\Longrightarrow$ anomalous $0^{--}$ state and also for $p = 2\pi/l$ states




states: $J^{P_t} = 0^+(\circ), 0^-(\bullet), 2^+(*), 2^-(+)$

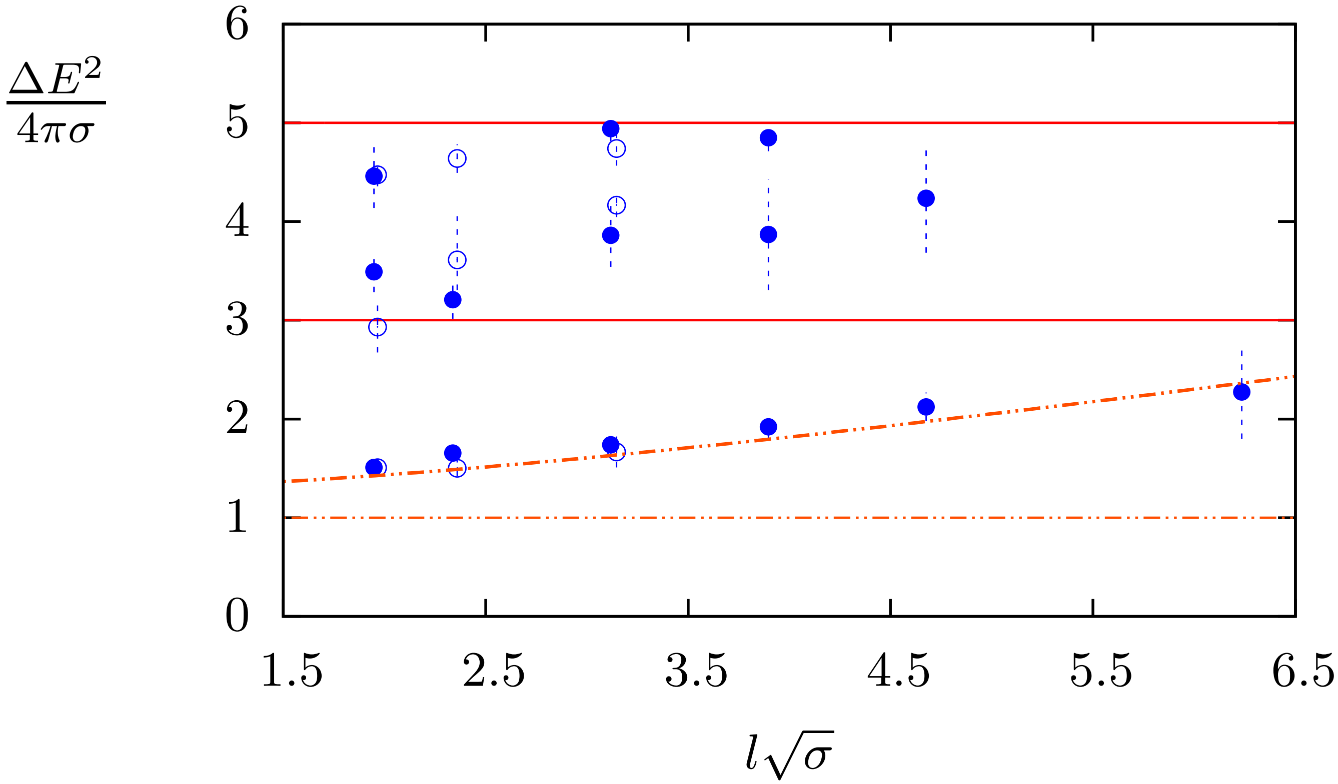
$\Longrightarrow$ anomalous $0^-$ state

$p = 0,\ 0^{--}$ : is this an extra state – is there also a stringy state?




ansatz: $E(l) = E_0(l) + m \qquad ; m = 1.85\sqrt{\sigma} \sim m_G/2$

similarly for $p=1,\ 0^-$ :   SU(3), $\bullet$; SU(5), $\circ$




ansatz: $E(l) = E_0(l) + (m^2 + p^2)^{1/2}$   ; $m = 1.85\sqrt{\sigma} \sim m_G/2$

BUT

Aharony, Klinghoffer arXiv:1008.2648

$\Rightarrow$

leading correction to Nambu-Goto in $D = 3 + 1$ is at $O(1/l^5)$ to excited states but not ground state
$\sim$ a 'spin-spin' interaction between right and left movers

Aharony, Komargodski, Schwimmer - in progress

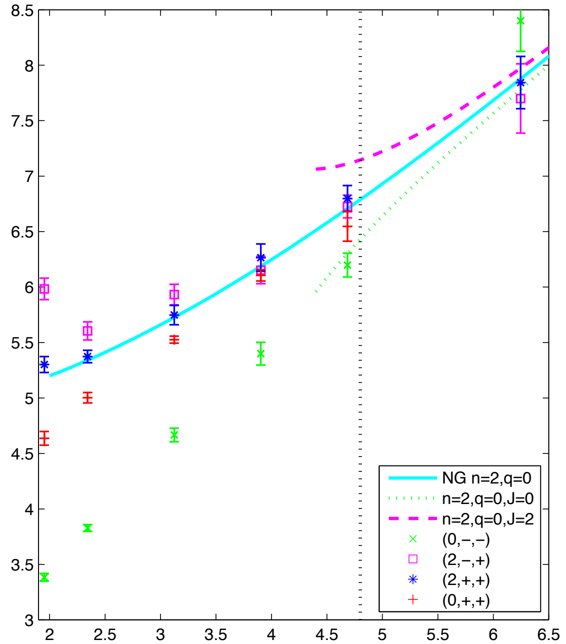
$\Rightarrow$

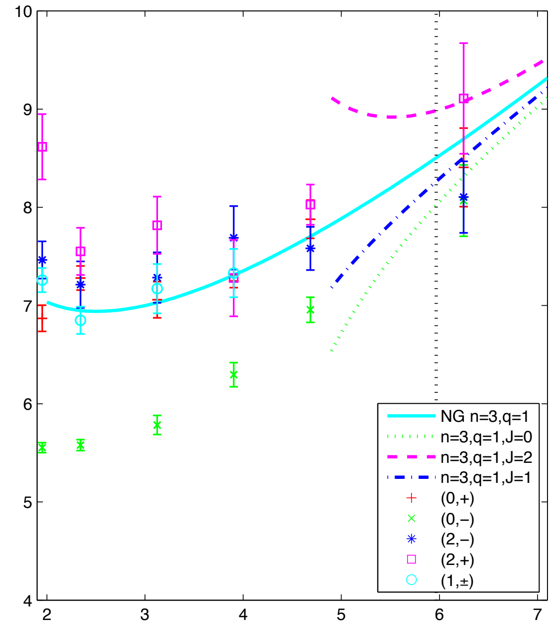
the value of the coefficient is *universal*

$$c_4 = \frac{(D-26)}{192\pi\sigma^2}$$

from Polchinski-Strominger rather than static-gauge

Aharony, Klinghoffer arXiv:1008.2648




The discrete points are the lattice results, the solid lines are the corresponding Nambu-Goto energy levels, and other lines include the shifts we calculated from using the specific value $c_4 = (D-26)/192\pi^2 T^2$. The vertical line is the expected radius of convergence for each level, we expect a matching only for points that are well to the right of this line.

fundamental flux $\longrightarrow$ higher representation flux

- $k$-strings: $f \otimes f \otimes ... k$ times, e.g.

$$\phi_{k=2A,S} = \frac{1}{2}\left(\{Tr_f\phi\}^2 \pm Tr_f\{\phi^2\}\right)$$

lightest flux tube for each $k \leq N/2$ is absolutely stable if $\sigma_k < k\sigma_f$ etc.

- binding energy $\Rightarrow$ mass scale $\Rightarrow$ massive modes?

- higher reps at fixed $k$, e.g. for $k = 1$ in SU(6)

$$f \otimes f \otimes \bar{f} \to f \oplus f \oplus \underline{84} \oplus \underline{120}$$

- $N \to \infty$ is not the 'ideal' limit that it is for fundamental flux:

– most 'ground states' are not stable (for larger $l$)

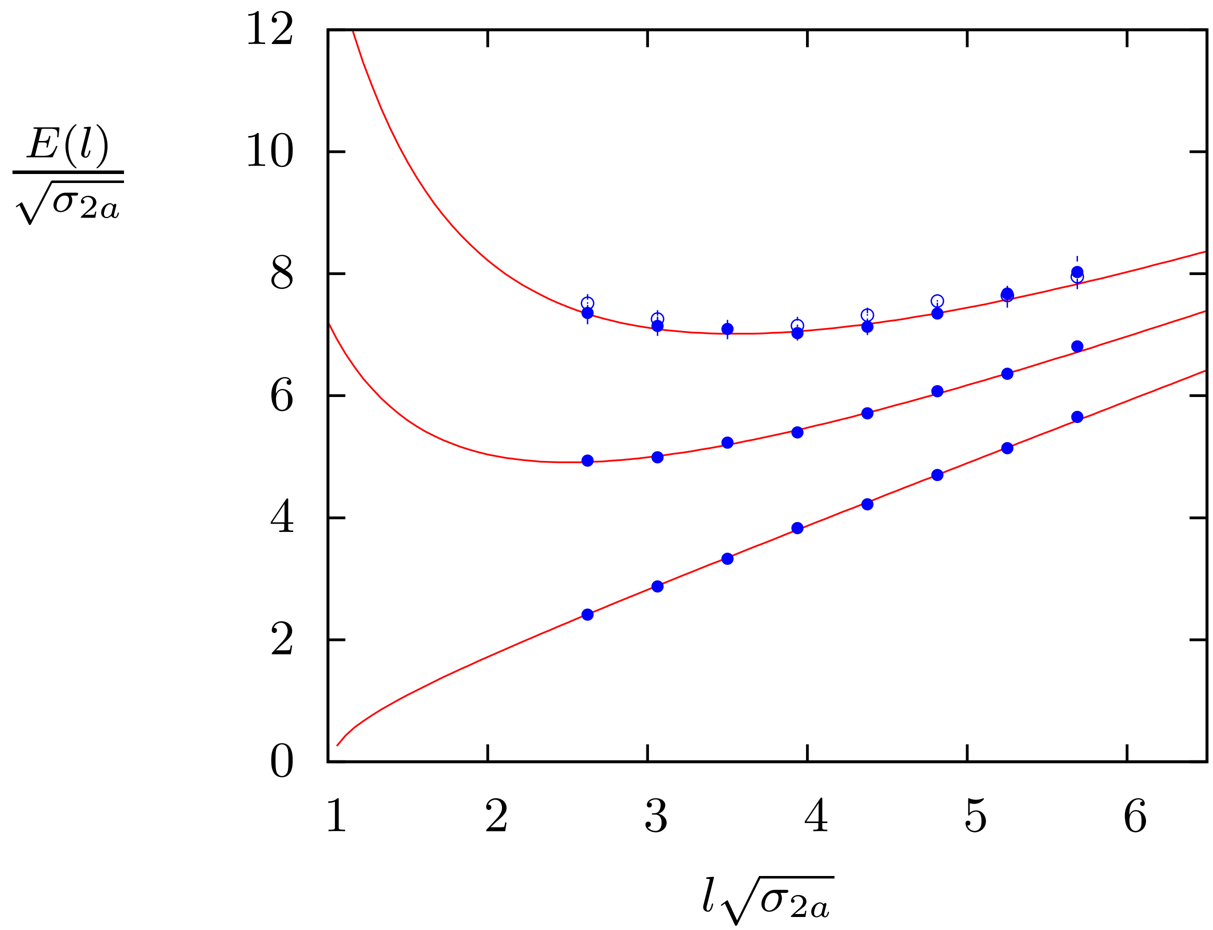
– typically become stable as $N \to \infty$, but

– $\sigma_k \to k\sigma_f$: states unbind?

$\longrightarrow$ some $D = 2 + 1$, SU(6) calculations ...

k=2A

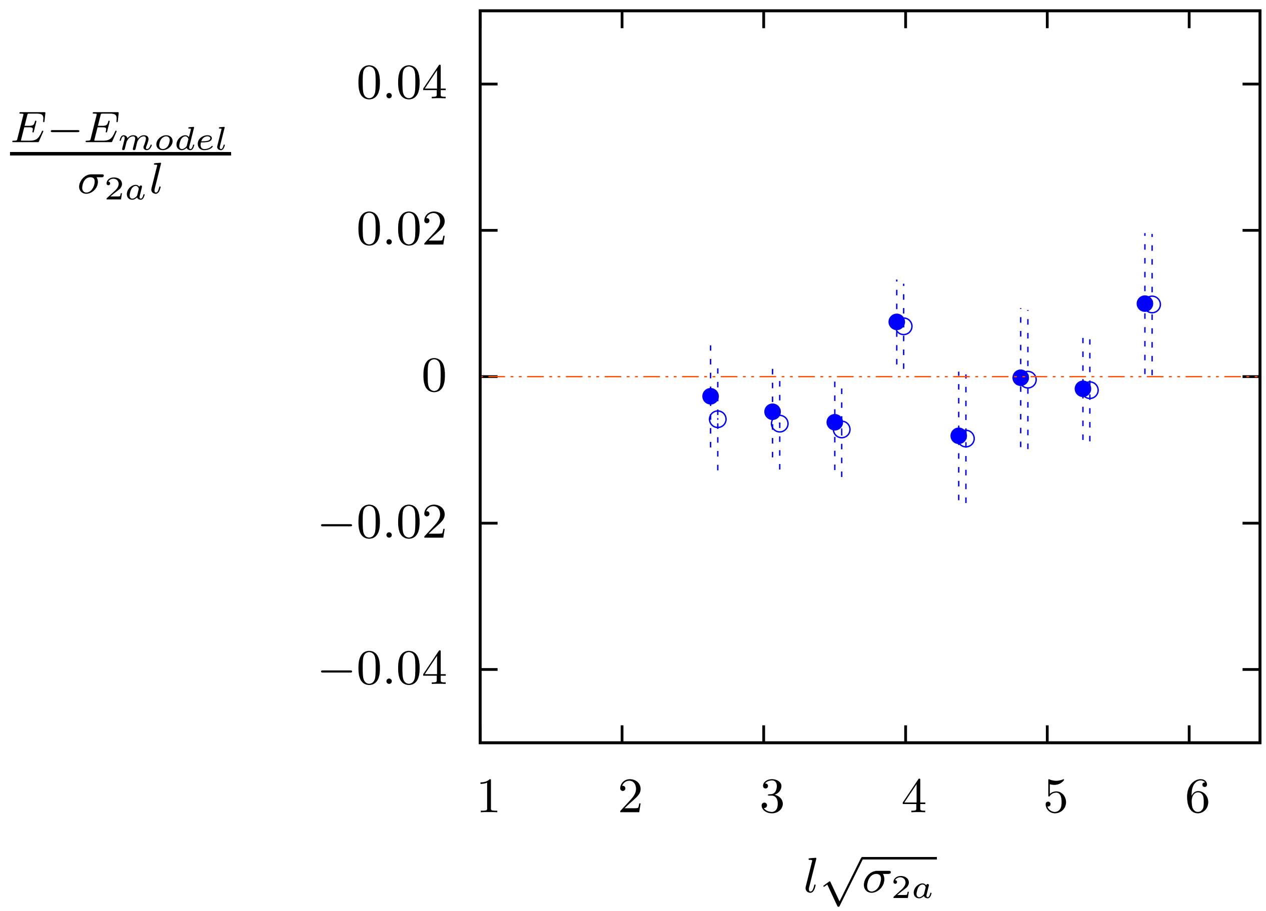
lightest $p = 2\pi q/l$ states with q=0,1,2



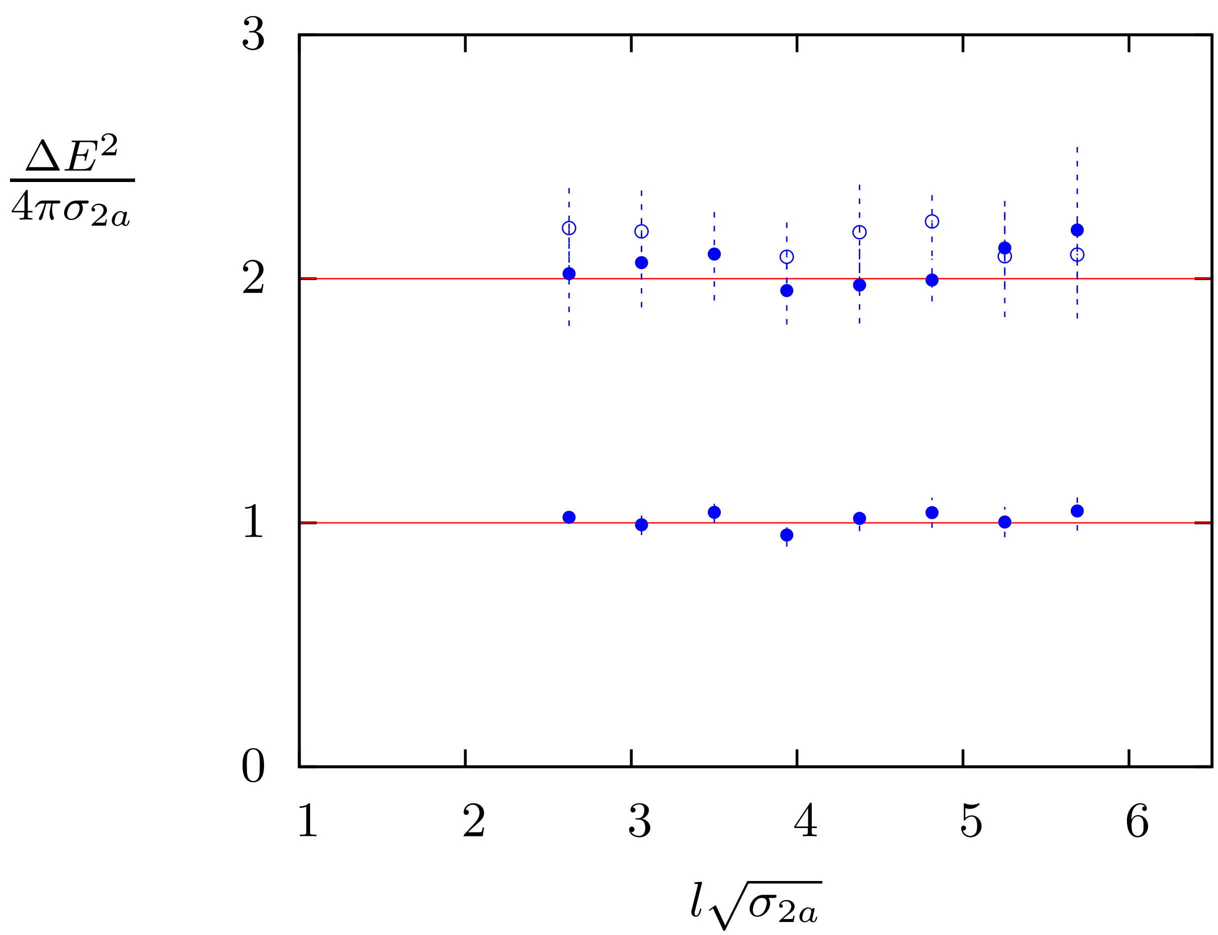

lines are NG

P=- (•), P=+ (∘)

k=2A ground state versus: Nambu-Goto (•), linear+Luscher (◦)

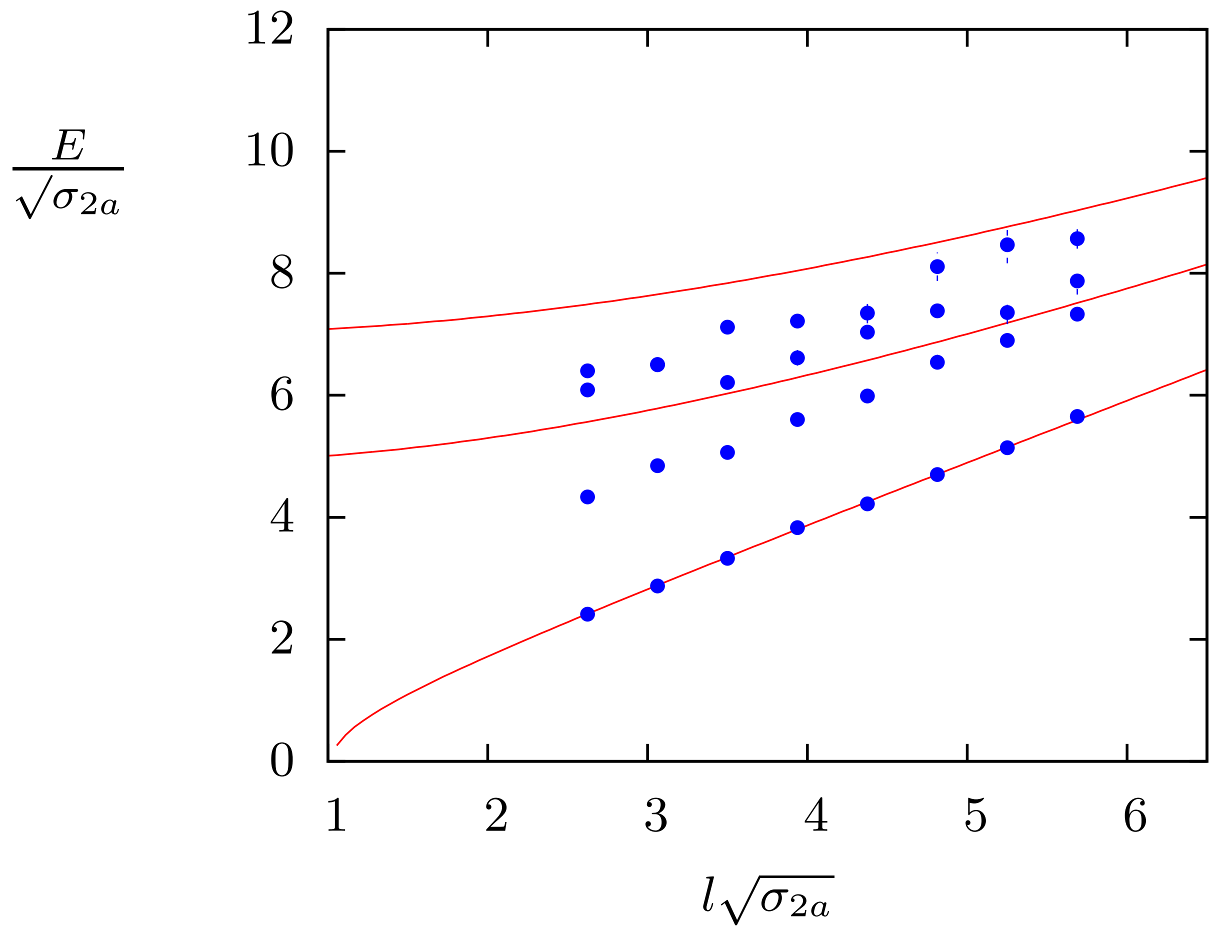



⇒ only sensitive to leading $1/l$ correction – but linear k=2A: versus Nambu-Goto, lightest $p = 2\pi/l,\ 4\pi/l$ states

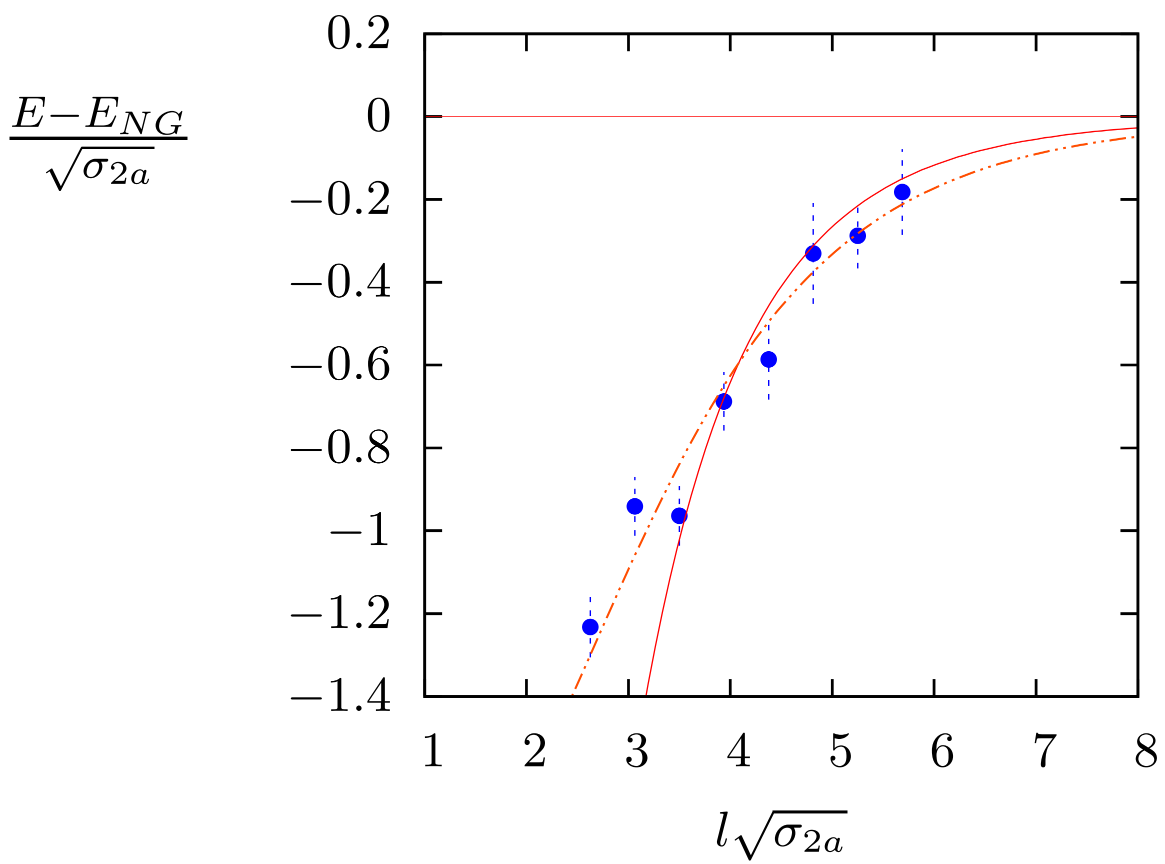



⇒ here very good evidence for NG

k=2A: lightest p=0, P=+ states



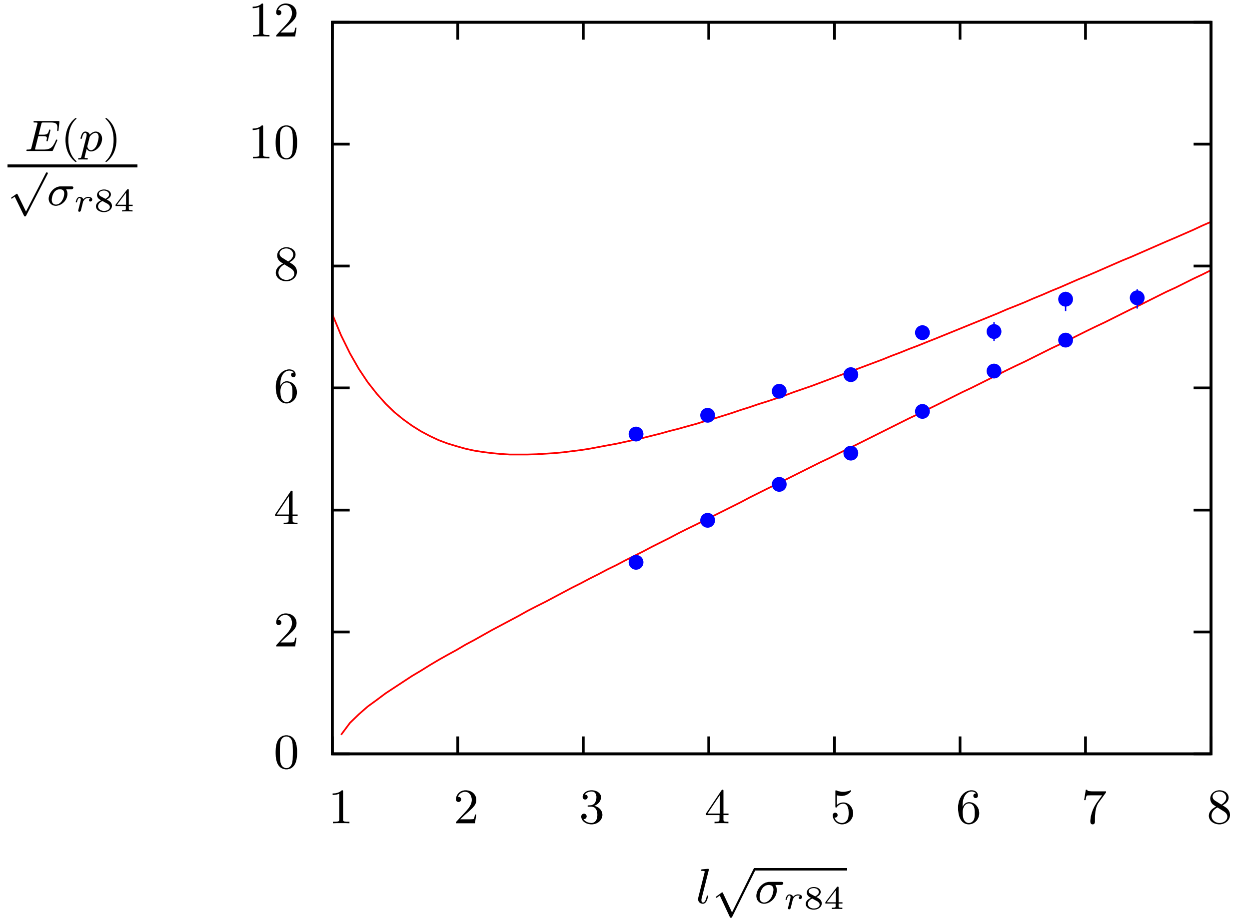

⇒ large deviations from Nambu-Goto for excited states k=2A: first excited p=0, P=+ state




⇒ deviations large ($\sim 10 c_{NG}$), but of 'typical' form:

$$\propto \frac{1}{l^7}\left(1+\frac{25}{l^2\sigma_{2a}}\right)^{-\gamma}, \quad \gamma = 2.75,\ 3.75$$

k=1, R=<u>84</u>: lightest $p = 0, 2\pi/l$ states




$\Rightarrow$ all reps come with Nambu-Goto towers of states

## Some conclusions on confining flux tubes and strings

- flux tubes are very like free Nambu-Goto strings, even when they are not much longer than they are wide

- this is so for all light states in $D = 2+1$ and most in $D = 3+1$

- ground state and states with one 'phonon' show corrections to NG only at *very* small $l$, consistent with $O(1/l^7)$

- most other excited states show small corrections to NG consistent with a resummed series starting with $O(1/l^7)$ and reasonable parameters

- in $D = 3+1$ we appear to see extra states consistent with the excitation of massive modes

• in $D = 2 + 1$, despite the much greater accuracy, we see no extra states

• we also find ‘towers’ of Nambu-Goto-like states for flux in other representations, even where flux tubes are not stable, but with much larger corrections – reflecting binding mass scale?

• theoretical analysis is complementary (in $l$) but moving forward rapidly, with possibility of resummation of universal terms and of identifying universal terms not seen in ‘static gauge’

there is indeed a great deal of simplicity in the behaviour of confining flux tubes and in their effective string description — much more than one would have imagined ten years ago ...