Buckling Phenomena in Fluids

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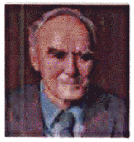

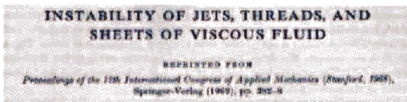

INSTABILITY OF JETS, THREADS, AND SHEETS OF VISCOUS FLUID

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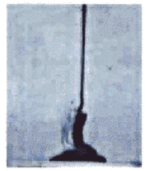
Proceedings of the 12th International Congress of Applied Mechanics (Stanford, 1968), Springer-Verlag (1969), pp. 382-8

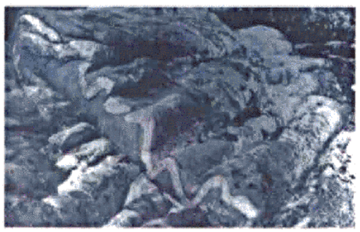
(1969)



$\sim 10^{-2}$ m

(M. Shin; 2000)

$\sim 10^{-6}$ m



~ 10 m

(Scotland; $-2.5 \cdot 10^{6}$!)

- What ? → Observations/experiments
- When ? How ? → Minimal theories
- Simple lessons ....

Geometry : Jets, threads & sheets ....

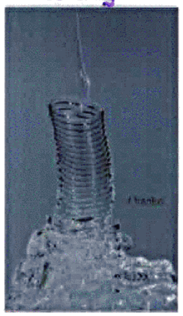
- Curves

Folding

2-d

~ 1 cm

Coiling

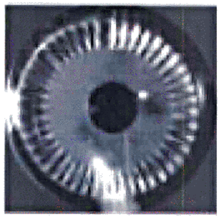


3-d

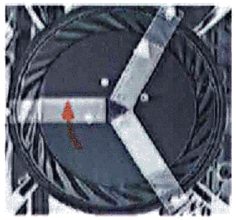
~ 1 cm

- Surfaces

Wrinkling



~ 3 cm



~ 8 cm

Common features:

- Inertialess flows
- Small aspect ratios + free surfaces
- Analogous to behaviour of elastic structures (filaments, sheets ...)

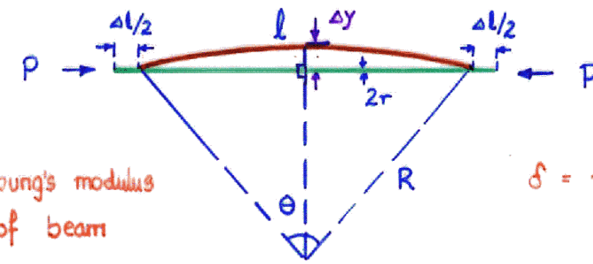
Stokes-Rayleigh analogy (~1860's)

	Static elasticity ≈	Creeping flow
	(small strain!)	(Inertialess, i.e. $Re = \rho U L/\mu << 1$)
	↓	↓
	displacement u	velocity $v \sim \partial u/\partial t$
	strain ∇u	strain rate $\nabla v \sim \frac{\partial}{\partial t}\nabla u$
Bulk:	Hookean solid	Newtonian fluid
	stress $\sigma \sim \nabla u$	stress $\sigma \sim \nabla v$
	· shear modulus G (Pa)	· shear viscosity μ (Pa.s)
	· Poisson ratio ν	· incompressibility ($\nu = 1/2$)
Surface:	traction-free (usually)	capillary forces, ... (usually)

⊕ Geometry ⇒ low-dimensional theories
(curves, surfaces)

Instabilities:

· Elastic buckling of a strut/beam — Euler (1740)

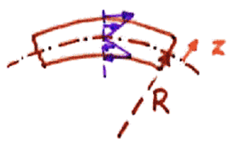


Y: Young's modulus of beam

$\delta = r/l << 1$

Geometry: $\theta << 1$ (at onset)

$$\therefore \quad \Delta l/l \sim l^2/R^2 \; ; \quad \Delta y/l^2 \sim 1/R$$

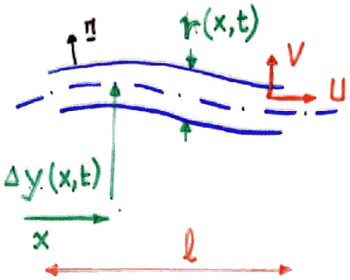
	Compression	Bending
		
strain	$\epsilon \sim \Delta l/l$	$\epsilon \sim z/R$; $\sigma \sim Yz/R$
stress	$\sigma \sim Y\Delta l/l$	
energy	$U_{compr.} \sim \int \sigma . \epsilon . dV \sim Y(\Delta l/l)^2 \cdot r^2 l$	$U_{bend.} \sim Y r^2/R^2 \cdot r^2 l$

Buckling ⇒ $U_{compr.} \sim U_{bend.}$ (exchange of stability)

$$\Rightarrow \Delta l/l \sim r^2/l^2 \sim \delta^2 << 1$$

$$\therefore \sigma_{crit.} \sim Y(\Delta l/l)_{crit.} \sim Yr^2/l^2 \; ; \quad P_{crit.} \sim Yr^4/l^2$$

· Viscous buckling ?



· $\delta = r/\ell \ll 1$

· small motions

i.e. $\Delta y \sim r$ . .

· $Re = \rho U L/\mu \ll 1$ (inertial / viscous)

· $Ca = \mu U/\gamma \gg 1$ (viscous / capillary)

⊕ bound. condn. $\underline{\sigma}\,\underline{n} = 0$; $\frac{D}{Dt}(H \pm h/2) = V$

Biot ; 1960's
(Buckmaster et al; 1970's)

Varicose

Sinuous

$1/R \sim \Delta y/\ell^2$

· $U/V \gg 1$

· $V/U \gg 1$?

strain rate $\frac{d\epsilon}{dt} = \epsilon_t \sim 1/\ell \cdot \overset{U}{(d\ell/dt)}$

$\epsilon_t \sim (z/R)_t \sim z/\ell^2 \cdot \overset{V}{(dy/dt)}$

stress resultant $\int \mu \epsilon_t \, dA \sim \mu \frac{r^2}{\ell} U$

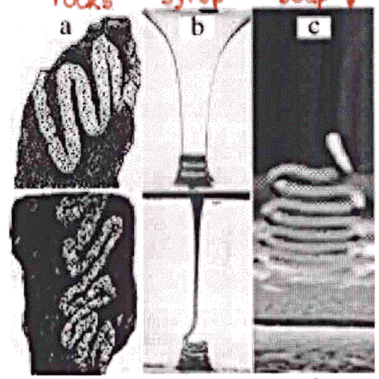
$\int \mu z/\ell^2 \, dA \, . V = 0$

torque balance → $\mu r^2 U/\ell \cdot \Delta y \sim \mu r^3 U/\ell \approx \mu r^4/\ell^2 \, V$

$\Rightarrow V/U \sim \ell/r \sim 1/\delta \gg 1$ · $\boxed{t_{bending}/t_{thickening} \sim \delta^2 \ll 1}$

FOLDING (M. Skorobogatiy, LM; 2000)

low Re # instability



A.T. Fomenko
(~1980)

Mechanism : gravitational time scale (extensional flow) $\overset{?}{\gtrless}$ viscous spreading time scale

film/jet
$r = r(H)$ - thickness
H = fall height

$\tau_g \sim \mu/\rho g H$

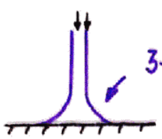
$\tau_\mu \sim \mu/\rho g r$

As $H \uparrow$, $r/H \downarrow \Rightarrow (\tau_g/\tau_\mu)^2 \sim r^2/H^2 \downarrow$

i.e. steady axisymmetric stagnation flow loses stability !

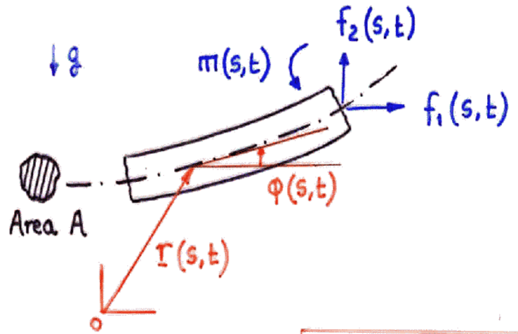
Linear stability : 'Approx.' ($1+\epsilon$ -dimensional) analysis

(Biot et al; 1960, Buckmaster et al. ; 1970's,
Cruickshank et al. ; 1980's , Yarin et al. 1990's)



3-d problem !

Nonlinear regime : Easier ! ← long-wavelength approx. !



s - "arc-length"
t - time

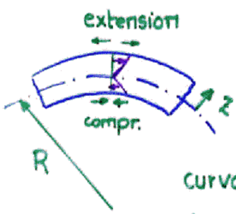
$(\cdot)_s = \partial(\cdot)/\partial s$

$(\cdot)_t = \partial(\cdot)/\partial t$

$$\boxed{\begin{array}{l} \underline{f}_s + A\rho\underline{g} = \underline{0} \\ m_s + \underline{r}_s \times \underline{f} = 0 \end{array}}$$

force balance
torque "

<u>Kinematics</u> : · Inextensible center line i.e. $\boxed{\begin{array}{l} x_s = \cos\phi \\ y_s = \sin\phi \end{array}}$



curvature
$\phi_s = 1/R$

· Bending deformations dominate

→ axial strain $\epsilon \sim z/R \sim z\phi_s$ elastic

" " <u>rate</u> $\epsilon_t \sim z\phi_{st}$ viscous

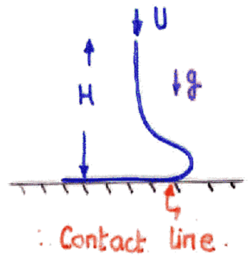
∴ Stress-resultant $\sim \int_{-r}^{r} \mu\,\epsilon_t\, dA = 0$!

But, Torque resultant $m(s,t) \sim \int_{-r}^{r} \mu z\,\epsilon_t\, dA \neq 0$

& $\boxed{m(s,t) = 3\mu \cdot \frac{\pi}{4} r^4 \cdot \phi_{st}}$ viscous

$= 3G\, \pi r^4/4 \cdot \phi_s$ elastic (incompr.)

∴ folding ≡ '1-d' problem for a dynamic 'contact line' as a fold is laid out.



$$\phi(s_i(t),t) = \begin{cases} 0 & \rightarrow \\ \pi & \leftarrow \end{cases}$$

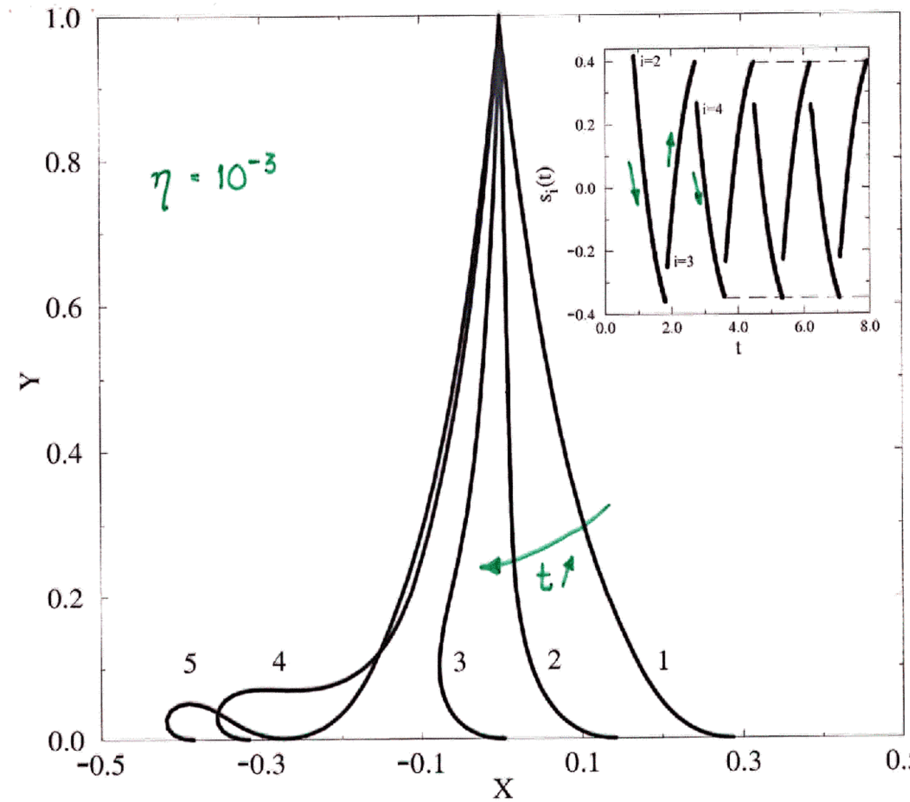
$$\phi_s(s_i(t),t) = 0$$

$$x(s_i(t),t) = s_i(t)$$

$$y(s_i(t),t) = 0$$

Parameter

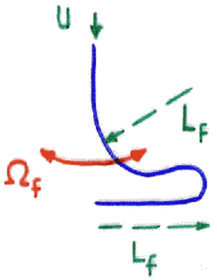
$$\eta = \frac{\mu r^2 U}{\rho g H^4} \ll 1$$



$\mu \sim 500$ Pa.s

$r/H \lesssim 10^{-1}$, $U \sim 1$ cm/s → $\eta \sim 10^{-3}$

$$\eta = \frac{r^2}{H^2}\cdot\frac{\mu U}{\rho g H^2} \ll 1 \quad ?$$



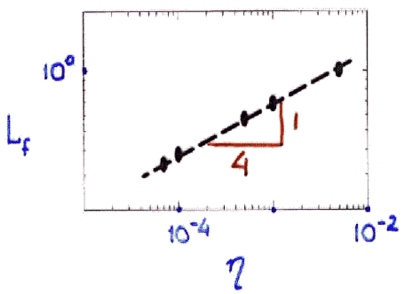
$f_1 \sim f_2 \sim \rho g r^2 L_f$ forces

torque balance ⇒ $\mu r^4 K_t \sim \rho g r^2 L_f\, L_f$

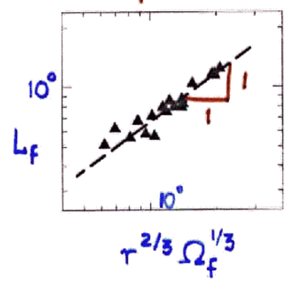
$K \sim 1/L_f$; $K_t \sim \Omega/L_f$

Periodic folds ⇒ $U \sim \Omega_f L_f$

Numerical sol'n.



Experiments in soap films

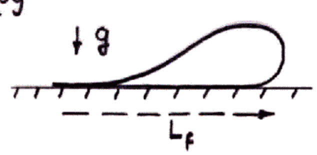


⇒ $L_f \sim \eta^{1/4} H \sim \left(\mu r^2 U/\rho g\right)^{1/4}$

$\Omega_f \sim \eta^{-1/4}\frac{U}{H} \sim \left(\mu r^2/\rho g U^3\right)^{-1/4}$

⇒ $L_f \sim \left(\frac{\mu\Omega_f r^2}{\rho g}\right)^{1/3}$

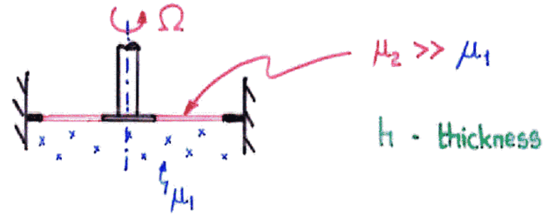
Elastic analog



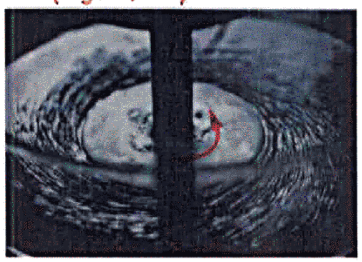
$\rho g r^2 L_f \cdot L_f \sim Y r^4 \cdot \frac{1}{L_f}$

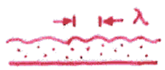
⇒ $L_f \sim \left(Y r^2/\rho g\right)^{1/3}$

WRINKLING

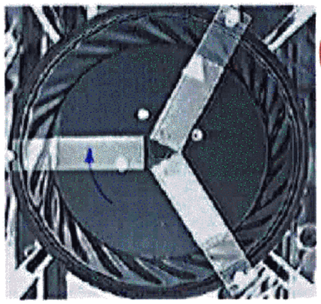


(Taylor; 1969)





$\lambda \ll h$

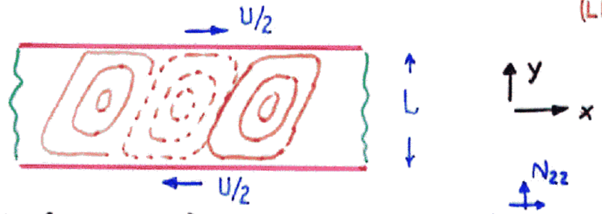


(Teichman, McKinley, LM) 2003

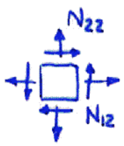
$\lambda \gg h$

model problem (LM; 2001)

thin-layer Couette flow

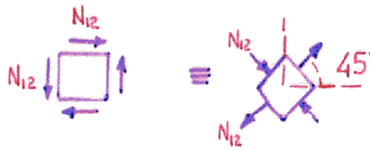


$w(x,y,t)$ - out-of-plane deformation



in-plane stress resultants

Onset of instability



$$\frac{\mu h^3}{3}\nabla^4 w_t = \cancel{N_{11}\, w_{,xx}} + \cancel{N_{22}\, w_{,yy}} + 2N_{12}\, w_{,xy}$$

$\because\ N_{11} = N_{22} = T$; $Ca = \mu U/T \gg 1$; $N_{12} = \mu\dot{\gamma}h$ (constant) control parameter

B.C. : • $w(x,y,t) = w_{,y}(x,y,t) = 0$ along $y = \pm L/2$

• periodic in x-direction

let $w = \frac{1}{2}\left[e^{\sigma t} f(y)\, e^{ikx/L} + c.c.\right]$; $\dot{\gamma} \rightarrow \dot{\gamma}/\sigma = S$

→ o.d.e. for $f(y)$ + 4 homog. b.c.

→ eigenvalue problem for $S, f(y)$!

Min. S → max. σ → selected mode shape !

(Southwell Skan ; 1920's)

elastic sheet

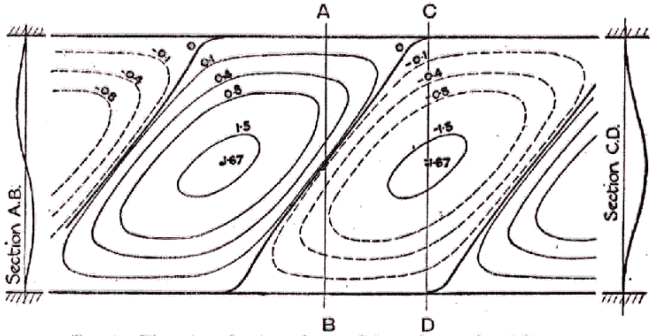
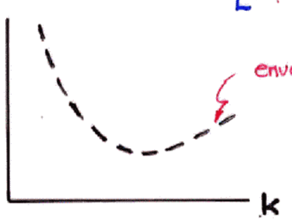


FIG. 4.—Distortion of strip under conditions of neutral stability.

• No threshold - viscous case (in the absence of • gravity • capillarity)

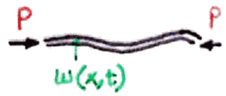
• Elastic case :- $Gh^3/L^4\, w \sim hN/L^2\, w \Rightarrow N_c \sim h^2/L^2\, G$



(J. Teichman, LM, 2002)

Gravity & Capillarity ?

Model problem : 1-d layer parallel compression



$\mu h^3 \omega_{xxxxt} - \gamma\,\omega_{xx} + Ph\,\omega_{xx} + \rho g\,\omega = 0$

bending / surface tension / Compr. / gravity

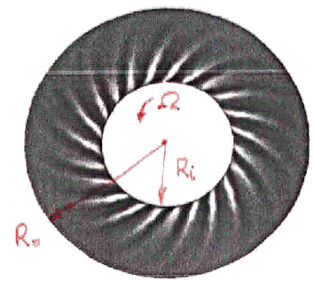
$\omega(x,t) = A e^{\sigma t + ikx}$ → $\mu h^3 \sigma = (Ph-\gamma)/k^2 - \rho g/k^4$

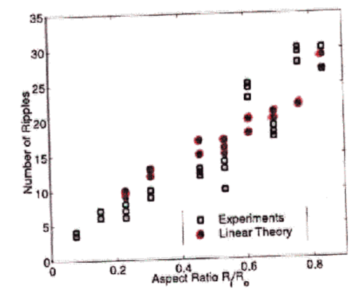
growth rate

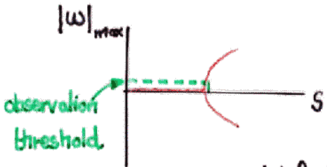
→ $Ph > \gamma$ for instability onset ! & $k \sim \left[\rho g/(Ph-\gamma)\right]^{1/2}$

Complete Couette problem : $\mu h^3 \nabla^4 \omega_t - \gamma \nabla^2 \omega + \rho g \omega = S/r^3\,(\omega_\theta/r - \omega_{r\theta})$

B.C. $\omega|_{r=R_o} = \omega|_{r=R_i} = 0$; $\partial\omega/\partial r|_{r=R_o,R_i} = 0$; $S \equiv \mu h \Omega R_i^2$







pitchfork bifurcation

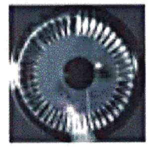
Nonlinear saturation : $|\omega|_{max} \sim \lambda$

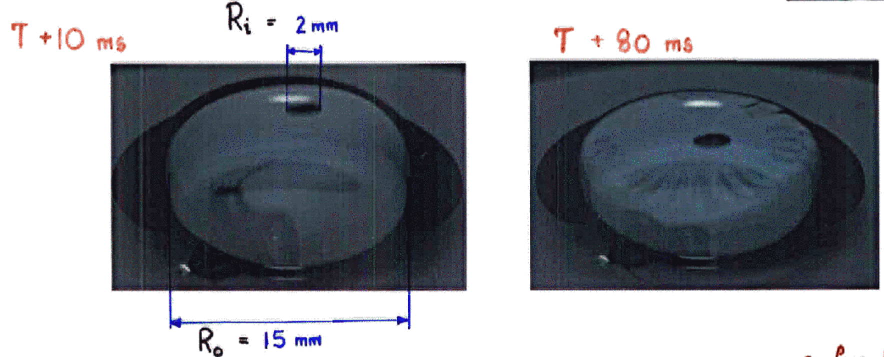
$\mu h \dot{\gamma} . \kappa \sim \rho g |\omega|_{max}$.

curvature

→ $|\omega|_{max} \sim (\mu h \dot{\gamma}/\rho g)^{1/2}$

WRINKLING (R. da Silveira, S. Chaieb, LM; 2000)





surface tension γ

· Dynamics of hole retraction in bursting films

(G I; 1960) Inertially limited ✗✗

Viscously limited ✓✓ (Debregeas et al; 1998)

$\mu \sim 10^3$ Pa.s

$\gamma \sim 20$ mN/m

$r \sim 1-10\ \mu m$

· Instabilities ?
- · rim ✗✗
- · surface ✓✓

· Mechanism: hole opens/retracts → air flow/pressure equalization → bubble collapse

$$\frac{t_{retraction}}{t_{collapse}} \sim \frac{\mu r/\gamma}{\sqrt{r/g}} \sim 10^4 >> 1$$

→ $R_i \sim$ const. (instability onset)

→ Flattening of a curved viscous shell ?
($r/R_o << 1$!)

Recall : $t_{bending}/t_{thickening/stretch} \sim (r/R_o)^2 << 1$ ✓

→ wrinkling transient precedes death !

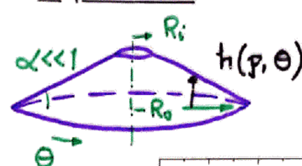
But geometry dictates how ...

∵ curved surface deformations <u>usually</u> couple bending + stretching at leading order !

e.g.  ∴ wrinkling → Inextensional (isometric) deformations <u>almost</u> <u>everywhere</u>

i.e. Gauss curvature is preserved a.e.

<u>Simple model</u>

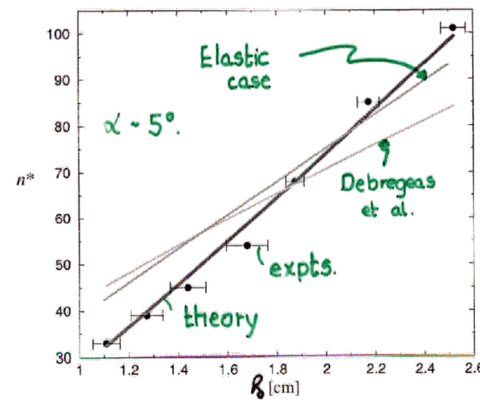


· const. thickness conical shell

· $h \to h + \delta h$; isometric ($ds^2 = d\bar{s}^2$)

· Elastic case : min. $\int$(Bending energy + Grav. pot. energy)

$$\pi^* = \left[\frac{\rho g R_i^3}{Y r^2}\frac{1}{\alpha}\,\psi\!\left(\frac{R_o}{R_i}\right) - \varphi\!\left(\frac{R_o}{R_i}\right)\right]^{1/2}$$



→ Threshold for instability (large ρ, R_i !)

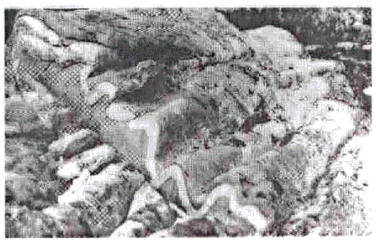
i.e. heavy cones with large holes will wrinkle !

· Viscous case : $Y \to \mu/3\tau$; $\tau \sim \sqrt{r/g} \sim 5$ ms

Where to ?

- Re $\not\ll$ 1 ?
- Ca $\not\gg$ 1 ?
- Rheology ?
- External fluid ?

electrosprays



geology

Summing up : geometry ⊕ Rayleigh analogy

i.e. · 'All you need is Love*, Love* is all you need'

(P. McCartney ~1960's)

* Love, A.E.H., 'Mathematical Treatise on Elasticity'...

· 'Tis a matter of time (scales) (Maxwell, 1867)

'In five minutes, you will say it is all so absurdly simple'

(S. Holmes ?)