2d Partition Functions, Elliptic Genera and Dualities

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with:		
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	D. Park, P. Zhao	(work in progress)

QFTs have a non-perturbative definition: path-integral

$$Z(t) = \int \mathcal{D}\Phi \, e^{-S[\Phi,t]}$$

Big progress on Euclidean path-integral of SUSY theories on compact manifolds

Technical tool: supersymmetric localization $\Rightarrow$ compute *exactly*

Also compute VEVs of SUSY operators: local and non-local, order and disorder

$$Z_{\mathcal{M}}(t, \mathcal{O}) = \int \mathcal{D}\Phi \, \mathcal{O} \, e^{-S[\Phi,t]}$$

Not new. [Witten 88]
New is connection with generic SUSY backgrounds (other than topological twist)

Various dimensions, amount of SUSY, compact manifolds, types of operators, ...

- Examples: S^d partition functions

S^5 with $\mathcal{N} = 1$ SUSY [Hosomici, Seong, Terashima 12; Kallen, Qiu, Zabzine 12; Kim, Kim 12]
S^4 with $\mathcal{N} = 2$ SUSY [Pestun 07]
S^3 with $\mathcal{N} = 2$ SUSY [Kupustin, Willett, Yaakov 09; Jafferis 10; Hama, Hosomichi, Lee 11]
S^2 with $\mathcal{N} = (2,2)$ SUSY [FB, Cremonesi 12; Doroud, Gomis, Le Floch, Lee 12; Doroud, Gomis 12]

- Generalizations: *e.g.* squashing of spheres

[Hama, Hosomichi, Lee 11; Imamura, Yokoyama 11; Hama, Hosomichi 12; Imamura 12]

- Other manifolds: *e.g.* $S^{d-1} \times S^1$

Index: $$I(f) = \mathrm{Tr}\,(-1)^F\, e^{-\beta H}\, f_i^{\mathcal{O}_i}$$

5d with $\mathcal{N} = 1$ SUSY [Kim, Kim, Lee 12]
4d with $\mathcal{N} = 1$ SUSY [Römelsberger 05; Gadde, Gaiotto, Pomoni, Rastelli, Razamat, Yan]
3d with $\mathcal{N} = 2$ SUSY [Kim 09; Imamura, Yokoyama 11]
2d with $\mathcal{N} = (0,2)$ SUSY [FB, Eager, Hori, Tachikawa 13; Gadde, Gukov 13]

- Ω-backgrounds [Nekrasov 02; Nekrasov, Okounkov 03; Shadchin 06]

Two-dimensional theories

Interesting for many reasons:

- interesting in their own right
- avatars of 4d theories (χ symmetry breaking, dyn. generated gap, ...)
- relevant for string theory
- directly connected with geometry, through non-linear sigma model (NLSM)
- connected to 4-manifolds with surface defect through M5-branes

[Gadde, Gukov, Putrov 13]

Partition functions

Exact evaluation of Euclidean path-integral on compact manifolds, useful for:

- Exact physical results (*e.g.* VEVs of operators)
- Precision tests of non-perturbative dualities
- Extract geometric information
  (Gromov-Witten invariants, elliptic genera, cluster algebra structures, ...)

Outline

- Localization and SUSY backgrounds
- S^2 partition function
- Elliptic genus (T^2 partition function, or 2d index)
- Non-perturbative dualities

Localization and SUSY backgrounds

Localization

Path-integral of Euclidean SUSY theory on $\mathcal{M}_d$:

$$Z_{\mathcal{M}_d}(t) = \int \mathcal{D}\Phi \; e^{-S[\Phi,t]}$$

Parameters t:

- from flat space Lagrangian
- controlling curvature couplings
- from curved metric on $\mathcal{M}_d$

With enough SUSY, *exactly* computable with localization techniques. [Witten 88, 91]

- Compute VEVs of SUSY operators as well:

$$Z_{S^d}(t,\mathcal{O}) = \int_{S^d} \mathcal{D}\Phi \; \mathcal{O} \; e^{-S[\Phi,t]}$$

Both local and non-local, both order and disorder.

Localization

- Action S and operators $\mathcal{O}$, supersymmetric w.r.t. supercharge $\mathcal{Q}$:

$$[\mathcal{Q}, S] = [\mathcal{Q}, \mathcal{O}] = 0$$

$\mathcal{Q}$-exact terms do not affect the path-integral:

$$\frac{\partial}{\partial u} \int \mathcal{D}\Phi \; \mathcal{O} \; e^{-S-u\,\{\mathcal{Q},\mathcal{P}\}} = 0$$

Z is sensitive only to $\mathcal{Q}$-cohomology (in space of functionals).

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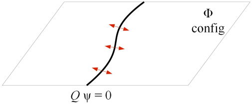
- Choose $\mathcal{Q}$-exact deformation action with positive definite bosonic part:

$$S_{\text{loc}} = u \sum_{\text{fermions } \psi} \mathcal{Q}\left((\overline{\mathcal{Q}\psi})\, \psi \right) \qquad\qquad S_{\text{loc}}\big|_{\text{bos}} = u \sum_{\psi} \big|\mathcal{Q}\psi\big|^2$$

$u \to \infty$ limit: only BPS configurations $\mathcal{Q}\psi = 0$ contribute

$$Z = \sum_{\Phi_0 \,|\, \mathcal{Q}\psi=0} e^{-S[\Phi_0]} \; Z_{\text{1-loop}}[\Phi_0]$$



Localization

Three tasks (after choosing $\mathcal{Q}$):

- Find space $\mathcal{M}_{\text{BPS}}$ of BPS configurations (must be finite dimensional!)
- Compute 1-loop determinant $Z_{\text{1-loop}}$
- Sum/integrate over $\mathcal{M}_{\text{BPS}}$

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What is new? SUSY backgrounds!

SUSY on curved manifolds

How do we preserve SUSY on a curved $\mathcal{M}_d$?

- Past: topological twist [Witten 88; Vafa, Witten 94; ...]

  Turn on background gauge field A_μ^R coupled to R-symmetry:

  $$``A_\mu^R = \omega_\mu"$$

  (embedding spin connection into R-symmetry)

  $\longrightarrow$ "scalar" supercharges are preserved

  This probes chiral / holomorphic sector.

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- Present: more general backgrounds [Pestun 07; ...]

  E.g.: probe real $=$ holomorphic $\times$ $\overline{\text{holomorphic}}$ sector

  Systematics explained by [Festuccia, Seiberg 11] [Adams, Jockers, Kumar, Lapan 11]

SUSY on curved manifolds

[Festuccia, Seiberg 11]

- Couple FT to external off-shell supergravity multiplet, turn on bosonic fields (including auxiliary) such that $\delta\psi_\alpha^\mu = 0$
- Take limit $G_N \to 0$ to decouple dynamical gravity but retain couplings to background.

$\delta\psi_\alpha^\mu = 0$	$\rightarrow$	generalized Killing spinor equation
$\delta^{\text{SUGRA}}(\text{matter})$	$\rightarrow$	$\delta^{\text{SUSY}}_{\text{curved}}(\text{matter})$
$\mathcal{L}^{\text{SUGRA}}$	$\rightarrow$	$\mathcal{L}^{\text{SUSY}}_{\text{curved}}$

This includes the topological twist, but gives *much more*!

- There exist *different* SUGRA multiplets

E.g. FZ multiplet, $\mathcal{R}$-multiplet, $\mathcal{S}$-multiplet $\longrightarrow$ different curved SUSYs

- SUSY algebra on $\mathcal{M}_d$ might be quite different from flat space

S^2 partition function

2d $\mathcal{N} = (2,2)$ SUSY with vector-like $U(1)_R$

FT:	$\mathcal{R}$-multiplet	$T_{\mu\nu}$,	S^{μ}_{α},	R^{μ},	J,	$\tilde{J}$	[Dumitrescu, Seiberg 11]
SUGRA:	"new minimal"	$g_{\mu\nu}$,	ψ^{α}_{μ},	A^{R}_{μ},	H,	$\tilde{H}$	

Killing spinor equations:

$$(\nabla_\mu - iA^R_\mu)\,\epsilon = -\tfrac{1}{2}H\gamma_\mu\epsilon - \tfrac{i}{2}\tilde{H}\gamma_\mu\gamma_3\epsilon$$

$$(\nabla_\mu + iA^R_\mu)\,\bar{\epsilon} = -\tfrac{1}{2}H\gamma_\mu\epsilon + \tfrac{i}{2}\tilde{H}\gamma_\mu\gamma_3\bar{\epsilon}$$

[Klare, Tomasiello, Zaffaroni 12; Closset, Dumitrescu, Festuccia, Komargodski 12]

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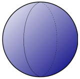
- $A^R_\mu = \pm\omega_\mu$, $H = \tilde{H} = 0$: topological twist (1/2 BPS)

A 

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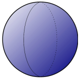
- $A^R_\mu = \pm\omega_\mu$, $H = \tilde{H} = 0$: topological twist (1/2 BPS)
- Round S^2: $A^R_\mu = 0$, $H = -\frac{i}{r}$, $\tilde{H} = 0$ no twist (1 BPS)

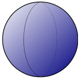
Killing spinors:

$$\nabla_\mu\epsilon = \frac{i}{2r}\gamma_\mu\epsilon \qquad \nabla_\mu\bar{\epsilon} = \frac{i}{2r}\gamma_\mu\bar{\epsilon}$$

[FB, Cremonesi 12]

[Doroud, Gomis, Le Floch, Lee 12]

A 

H 

2d $\mathcal{N} = (2,2)$ SUSY with vector-like $U(1)_R$

- Two-dimensional $\mathcal{N} = (2,2)$ theories with a vector-like $U(1)_R$ R-symmetry can be placed supersymmetrically on S^2 (2 complex supercharges)

Superalgebra: $\mathfrak{su}(2|1) \quad \supset \quad \mathfrak{su}(2) \times \mathfrak{u}(1)_R$

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}] = \mathcal{L}_\xi^A + \frac{i}{2r}\alpha R \qquad\qquad \xi^\mu = i\bar{\epsilon}\gamma^\mu\epsilon$$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = [\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}] = 0 \qquad\qquad \alpha = i\bar{\epsilon}\epsilon$$

- Vector multiplet: $V = (A_\mu, \lambda, \bar{\lambda}, \sigma + i\eta, D)$

  Chiral multiplet: $\Phi = (\phi, \bar{\phi}, \psi, \bar{\psi}, F, \bar{F})$

On S^2 freedom to choose R-charges $R[\Phi]$ of chiral multiplets $\quad \rightarrow \quad$ couplings

Class of theories

Gauge theories of vector and chiral multiplets

Actions:

- kinetic terms – $\mathcal{Q}$-exact (no dependence on gauge couplings)
- superpotential W – $\mathcal{Q}$-exact (dependence on R-charges!)
- twisted superpotential $\mathcal{W}$ (includes cplx FI term) – full dependence
- masses and ext fluxes (ext vector multiplets) – full dependence

Include Landau-Ginzburg models

At low energy: realize NLSM! (Käher and CY)

(Coulomb branch) localization formula

The S^2 partition function is: [FB, Cremonesi 12; Doroud, Gomis, Le Floch, Lee 12]

$$Z_{S^2} = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m}} \int \Big(\prod_j \frac{d\sigma_j}{2\pi} \Big) \, Z_{\text{class}}(\sigma, \mathfrak{m}) \, Z_{\text{gauge}}(\sigma, \mathfrak{m}) \, \prod_\Phi Z_\Phi(\sigma, \mathfrak{m}; M, \mathfrak{n})$$

The one-loop determinants are:

$$Z_{\text{gauge}} = \prod_{\alpha \in G \,,\, \alpha > 0} \Big(\frac{\alpha(\mathfrak{m})^2}{4} + \alpha(\sigma)^2 \Big)$$

$$Z_\Phi = \prod_{\rho \in R_\Phi} \frac{\Gamma\Big(\frac{R[\Phi]}{2} - i\rho(\sigma) - i f^a[\Phi] M_a - \frac{\rho(\mathfrak{m}) + f^a[\Phi] n_a}{2} \Big)}{\Gamma\Big(1 - \frac{R[\Phi]}{2} + i\rho(\sigma) + i f^a[\Phi] M_a - \frac{\rho(\mathfrak{m}) + f^a[\Phi] n_a}{2} \Big)}$$

The classical action is:

$$Z_{\text{class}} = e^{-4\pi i \xi \, \mathrm{Tr}\, \sigma - i\theta \, \mathrm{Tr}\, \mathfrak{m}} \exp \Big\{ 8\pi i r \, \mathbb{R}\mathrm{e} \, \widetilde{W} \big(\tfrac{\sigma}{r} + i \tfrac{\mathfrak{m}}{2r} \big) \Big\}$$

We isolated the linear piece in $\widetilde{W}$ (Fayet-Iliopoulos term)

What is it good for?

- Precision tests of dualities
  - Seiberg-like: $U(N_c)$ with N_f fund $\leftrightarrow$ $U(N_f - N_c)$ with N_f fund
  - Mirror symmetry (Hori-Vafa): gauge theory with charged matter $\leftrightarrow$ gauge theory axially coupled ($\widetilde{W}$) to neutral LG model

    [Gomis, Lee 12]
  - AGT: S^2-partition function $\leftrightarrow$ Liouville correlators with degenerate fields

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- VEVs of operators (*e.g.* Wilson line operators)

$$Z_{S^2}(\text{loop}) = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m}} \int \Big(\prod_j \frac{d\sigma_j}{2\pi} \Big) \ \mathrm{Tr}(e^{2\pi\sigma - i\pi\mathfrak{m}}) \ Z_{\text{class}} \ Z_{\text{1-loop}}$$

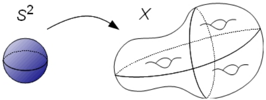
What is it good for?

- Geometry of Kähler moduli space, (equivariant) GW invariants of CYs:

$$Z_{S^2} = \langle \bar{0} | 0 \rangle_{RR} = e^{-K_{\text{Kähler}}}$$

[Jockers, Kumar, Lapan, Morrison, Romo 12]

[Bonelli, Sciarappa, Tanzini, Vasko 13]



Calabi-Yau 3-fold:

$$e^{-K_{\text{Kähler}}(t,\bar{t})} = -\frac{i}{6}\kappa_{lmn}(t^l - \bar{t}^l)(t^m - \bar{t}^m)(t^n - \bar{t}^n) + \frac{\zeta(3)}{4\pi^3}\chi(Y_3) + \mathcal{O}(e^{2\pi i t})$$

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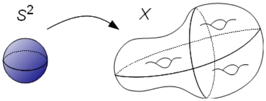

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- Central charges of D-branes (D_2 partition function)

[Hori, Romo 13; Honda, Okuda 13; Kim, Lee, Yi 13]

Elliptic genera

Definition

- Hamiltonian definition (index, only $H_R = 0$ states):

with $\mathcal{N} = (2,2)$:
$$\mathcal{I}(\tau, z, u_a) = \mathrm{Tr}_{\mathsf{RR}}(-1)^F q^{H_L} \bar{q}^{H_R} y^{J_L} \prod_a x_a^{K_a}$$

with $\mathcal{N} = (0,2)$:
$$\mathcal{I}(\tau, u_a) = \mathrm{Tr}_{\mathsf{RR}}(-1)^F q^{H_L} \bar{q}^{H_R} \prod_a x_a^{K_a}$$

Parameters: $q = e^{2\pi i \tau}$, $y = e^{2\pi i z}$, $x_a = e^{2\pi i u_a}$

Superconformal theory: $H_L = L_0 - \frac{c_L}{24}$, $H_R = \bar{L}_0 - \frac{c_R}{24}$, $J_L = J_0$

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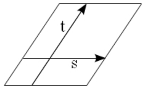
- Lagrangian definition:

path integral on T^2 with ext flat connections



$$z = \oint_{\mathsf{t}} A^{\mathsf{R}} - \tau \oint_{\mathsf{s}} A^{\mathsf{R}} , \qquad u_a = \oint_{\mathsf{t}} A^{a\text{-th flavor}} - \tau \oint_{\mathsf{s}} A^{a\text{-th flavor}}$$

Definition

- Geometric definition for NLSM on M — case $\mathcal{N} = (2,2)$:

$$\mathbb{E}_{q,y} = \bigotimes_{n \geq 1} \left[\bigwedge\nolimits^{\bullet}_{-y^{-1}q^n} T_M \otimes \bigwedge\nolimits^{\bullet}_{-yq^{n-1}} T^*_M \otimes S^{\bullet}_{q^n}(T_M \otimes T^*_M) \right]$$

where $\quad \bigwedge\nolimits^{\bullet}_t V = \sum_{i=0}^{\infty} t^i \bigwedge\nolimits^i V \quad$ and $\quad S^{\bullet}_t V = \sum_{i=0}^{\infty} t^i S^i V$.

Holomorphic Euler characteristic (Hirzebruch-Riemann-Roch):

$$\chi(M;\tau,z) = y^{-\frac{d}{2}} \int_M \mathsf{ch}(\mathbb{E}_{q,y})\,\mathsf{Td}(M) = \int_M \prod_{j=1}^{d} \frac{\theta_1(\tau|\xi_j - z)}{\theta_1(\tau|\xi_j)}\,\xi_j$$

Elliptic genus

- Physics:

  information about the spectrum of the theory

- Mathematics:

  - information about the elliptic cohomology of target
  - provide examples of modular forms

Elliptic genus

We use Lagrangian definition: $\boxed{Z_{T^2}(\tau, z, u_a)}$

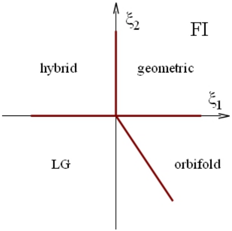
- $\mathcal{N} = (2,2)$ and $\mathcal{N} = (0,2)$ gauge theories of

vector + chiral (+ Fermi) multiplets:

- All action terms are $\mathcal{Q}$-exact!
  Expected: it is a supersymmetric index

- Dependence on ext flat connections (R-symmetry, flavor)

Phases of 2d gauge theories



Different "phases" as we vary FI:

- Classic example [Witten 93] : the quintic

$$U(1) \text{ gauge theory} + \text{chirals} + W = P\, f(X_{1\ldots5})$$

$\xi \ll -1$ → LG $\mathbb{Z}_5$-orbifold

$\xi \gg 1$ → quintic CY

In general: symplicial cones, secondary fan [Aspinwall, Greene, Morrison 93]

- Elliptic genus does not depend on FI

$\Rightarrow$ Gauge theory formula should unify known formulas

1) BPS space

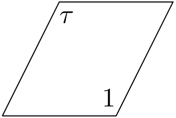
- Flat gauge connections (modulo gauge trans.):

$$\mathcal{M}_{\mathsf{BPS}} = \{A_\mu | F_{\mu\nu} = 0\}$$

Flat flavor and R-symmetry connections are fixed!

- With Abelian and simply-connected factors:

$$\mathcal{M}_{\mathsf{BPS}} = \mathfrak{M}/W \qquad \mathfrak{M} = \mathfrak{h}_{\mathbb{C}}/(\Gamma + \tau\Gamma) \simeq T^{2r}$$



2) 1-loop: Matter sector

Easy in Hamiltonian formulation — case $\mathcal{N} = (2,2)$:

Chiral multiplet :

	ϕ	ψ_R	ψ_L
J_L	$\frac{R}{2}$	$\frac{R}{2}$	$\frac{R}{2} - 1$
K	Q	Q	Q

Putting everything together:

$$\boxed{Z_{\Phi,Q}(\tau,z,u) = \frac{\theta_1(q, y^{\frac{R}{2}-1}x^Q)}{\theta_1(q, y^{\frac{R}{2}}x^Q)}}$$

in terms of the Jacobi theta function

$$\theta_1(q,y) = -iq^{\frac{1}{8}}y^{\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^{n-1})$$

Can also be written as plethystic exponential.

2) 1-loop: Matter sector

- $Z_{\Phi,Q}$ all we need for Landau-Ginzburg models [Witten 93]

W fixes R-charges

E.g.: A-series $\mathcal{N} = (2,2)$ minimal models: $W = \Phi^k$, $R = \frac{2}{k}$.

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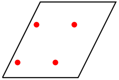
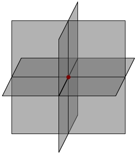
- Bosonic zero-modes for special values of the flat connections:

$$\frac{R}{2} z + Qu = 0 \pmod{\mathbb{Z} + \tau\mathbb{Z}}$$

Rank 1: poles on the torus



Higher rank: singular hyperplanes on the Jacobian $\mathfrak{M}$



Part of the symmetry is gauged $\Rightarrow$ potential problem

2) 1-loop: Gauge sector

Vector multiplet:

	σ	λ_R	λ_L
J_L	-1	-1	0
G_a	α	α	α

Vectors in the Cartan: LM gaugino has fermionic zero-mode! Removing it:

$$\boxed{Z_{V,G}(\tau,z,u) = \left(\frac{2\pi\eta(q)^3}{\theta_1(q,y^{-1})}\right)^{\mathrm{rank}\,G} \prod_{\alpha\in G} \frac{\theta_1(q,x^\alpha)}{\theta_1(q,y^{-1}x^\alpha)} \prod_{a=1}^{\mathrm{rank}\,G} du_a}$$

Define:

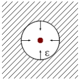
$$\boxed{Z_{\text{1-loop}}(\tau,z,u_a,\xi_a) = Z_{V,G}\prod_i Z_{\Phi_i}}$$

3) Integration

We should integrate over $\mathfrak{M}$: poles of $Z_{\Phi,Q}$!

Artifact of $e = 0$: poles are smoothed out at finite e (D-terms)

- Work at finite e (approx.) and with a cutoff ε: cut tubular regions around singularities

  Eventually take scaling limit $e \to 0$, $\varepsilon \to 0$.

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$$\eta \in \mathfrak{h}^*$$

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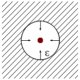
  

  Eventually take scaling limit $e \to 0$, $\varepsilon \to 0$.

- Getting smooth limit requires choice of regularization parameter:

$$\eta \in \mathfrak{h}^*$$

- Absorb fermionic zero-modes. Integral over $\mathfrak{M} \setminus \Delta_\varepsilon$ becomes contour integral:

$$Z_{S^2} = \int_{\mathcal{C}(\eta)} Z_{\text{1-loop}}(u)\, du$$

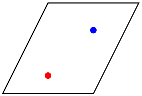
cfr. [Grassi, Policastro, Scheidegger 07]

- Result independent of η. Expression depends on ray of η.

Rank one

For $U(1)$: moduli space of flat connections

$$\mathfrak{M} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) = T^2$$



Divide singularities into "positive" and "negative" poles:

$$\mathfrak{M}_{\mathsf{sing}} = \mathfrak{M}^+_{\mathsf{sing}} \sqcup \mathfrak{M}^-_{\mathsf{sing}}$$

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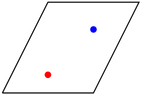


Divide singularities into "positive" and "negative" poles:

$$\mathfrak{M}_{\text{sing}} = \mathfrak{M}^+_{\text{sing}} \sqcup \mathfrak{M}^-_{\text{sing}}$$

Formula:

$$\begin{aligned} Z_{T^2}(\tau, z, \xi_a) &= \frac{1}{|W|} \sum_{u_* \in \mathfrak{M}^+_{\text{sing}}} \operatorname*{Res}_{u=u_*} Z_{\text{1-loop}}(\tau, z, u, \xi_a) \\ &= -\frac{1}{|W|} \sum_{u_* \in \mathfrak{M}^-_{\text{sing}}} \operatorname*{Res}_{u=u_*} Z_{\text{1-loop}}(\tau, z, u, \xi_a) \end{aligned}$$

- Two expressions come from $\eta = \pm$, and agree
- $U(1)$: two expr's correspond to large $\pm$ FI term $\to$ CY/LG correspondence
- Geometric phase: agrees with geometric definition (characteristic class)

Example: the quintic

Example of CY/LG correspondence

$U(1)$ gauge theory $\quad + \quad$ chirals $X_{1,\ldots,5},\ P \quad + \quad W = P\, f_5(X_{1,\ldots,5})$

- Positive poles:

$$Z_{T^2}(\tau,z) = \frac{i\eta(q)^3}{\theta_1(\tau|z)} \oint_{u=0} du\, \frac{\theta_1(\tau|-5u)}{\theta_1(\tau|z-5u)} \left[\frac{\theta_1(\tau|u-z)}{\theta_1(\tau|u)}\right]^5$$

  Agrees with geometric formula for quintic CY_3.

- Negative poles:

$$Z_{T^2}(\tau,z) = \frac{1}{5}\sum_{k,l=0}^{4} e^{-2\pi i z} \left[\frac{\theta_1(\tau|\frac{-4z+k+l\tau}{5})}{\theta_1(\tau|\frac{z+k+l\tau}{5})}\right]^5$$

  Landau-Ginzburg $\mathbb{Z}_5$-orbifold.

Higher rank

Space of flat connections: $\mathfrak{M}/W$ with $\mathfrak{M} = \mathfrak{h}_{\mathbb{C}}/(\Gamma + \tau\Gamma) \simeq T^{2r}$

Singular hyperplanes:

$$H_i = \{u \,|\, Q_i(u) + \text{shift} = 0 \pmod{\Gamma + \tau\Gamma}\}$$

Isolated intersection points: $\mathfrak{M}^*_{\text{sing}}$

Integration contour is specified by the Jeffrey-Kirwan residue:

$$\boxed{Z_{T^2}(\tau, z, \xi) = \frac{1}{|W|} \sum_{u_* \in \mathfrak{M}^*_{\text{sing}}} \underset{u=u_*}{\text{JK-Res}}\big(\mathsf{Q}(u_*), \eta\big) \;\; Z_{\text{1-loop}}(\tau, z, u, \xi)}$$

Depends on a choice of vector (ray) $\eta \in \mathfrak{h}^*$

Definition:

$$\underset{u=0}{\text{JK-Res}}(\mathsf{Q}_*, \eta) \, \frac{dQ_{j_1}(u)}{Q_{j_1}(u)} \wedge \cdots \wedge \frac{dQ_{j_r}(u)}{Q_{j_r}(u)} = \begin{cases} \operatorname{sign}\det(Q_{j_1} \ldots Q_{j_r}) & \text{if } \eta \in \mathrm{Cone}(Q_{j_1} \ldots Q_{j_r}) \\ 0 & \text{otherwise} \end{cases}$$

Non-perturbative dualities

Hori-Tong duality

[Hori, Tong 06]

$SU(k)$ with N fundamentals $\leftrightarrow$ $SU(N-k)$ with N fundamentals

Proved that:

- S^2 partition function agrees [FB, Cremonesi 12; Doroud, Gomis, Le Floch, Lee 12]
- elliptic genus agrees [FB, Eager, Hori, Tachikawa 13; Gadde, Gukov 13]

as functions of $U(N)$ flavor symmetry parameters.

Seiberg-like dualities

More general Seiberg-like dualities: (Grassmannian dualities)

$U(k)$ with N_f, N_a $\leftrightarrow$ $U\big(\max(N_f, N_a) - k\big)$ with N_a, N_f, singlets and $W = \tilde{Q}MQ$

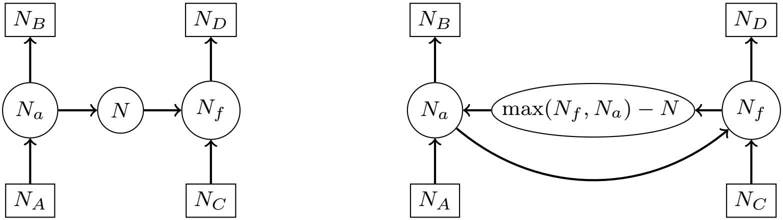
Proved that S^2 partition function and elliptic genus agree.

$Z_{S^2} \to$ precise map of parameters: *e.g.* FI term
contact terms in twisted superpotential $\mathcal{W}$

Quivers and cluster algebra

[FB, Park, Zhao, in progress]

When applied to quivers: FI's transform as cluster algebra! [Fomin, Zelevinsky 01]



FI transformation rules:

$$t'_j = \begin{cases} t_k^{-1} & \text{if } j = k \\ t_j t_k^{[b_{kj}]_+} (1+t_k)^{-b_{kj}} & \text{otherwise} \end{cases}$$

Quiver gauge theory $\rightarrow$ CY manifold $\rightarrow$ quantum Kähler moduli space

Ring of holomorphic functions on $\mathcal{M}_{\text{Kähler}}$ is a cluster algebra.

Implications for integrable systems, Pichard-Fucks equations, singularity theory…

Conclusions

Some interesting directions:

- Higher genus and lower supersymmetry
- Generalized Kähler geometry
- More general dualities (*e.g.* Kutasov-Schwimmer-like)
- Consequences for integrable systems

Thank you!