# The Helical Luttinger Liquid and the Edge of Quantum Spin Hall Systems

Congjun Wu

Department of Physics, UCSD

Ref: Phys. Rev. Lett. **96**, 106401 (2006).

with B. Andrei Bernevig (Princeton), Shou-Cheng Zhang (Stanford).

KITP workshop Dec 9, 2008

# Introduction

• Quantum spin Hall insulators: gapped bulk and gapless edge modes.

F. D. M. Haldane, PRL 61, 2015 (1988); Kane and Mele, Phys. Rev. Lett. 95, 146802 (2005), Phys. Rev. Lett. 95 225801 (2005). B. A. Bernevig et and Shou-cheng Zhang, PRL 96, 106802 (2006); X. L Qi et al., cond-mat/0505308; L. Sheng et al., PRL 95, 136602 (2005); D. N. Sheng et al, cond-mat/0603054.

M. Koenig, S. Widmann, C. Bruene, A. Roth, H. Buhmann, L. Molenkamp, X. Qi and S. C. Zhang Science 2007. D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, M. Z. Hasan, Nature 453 (2008).

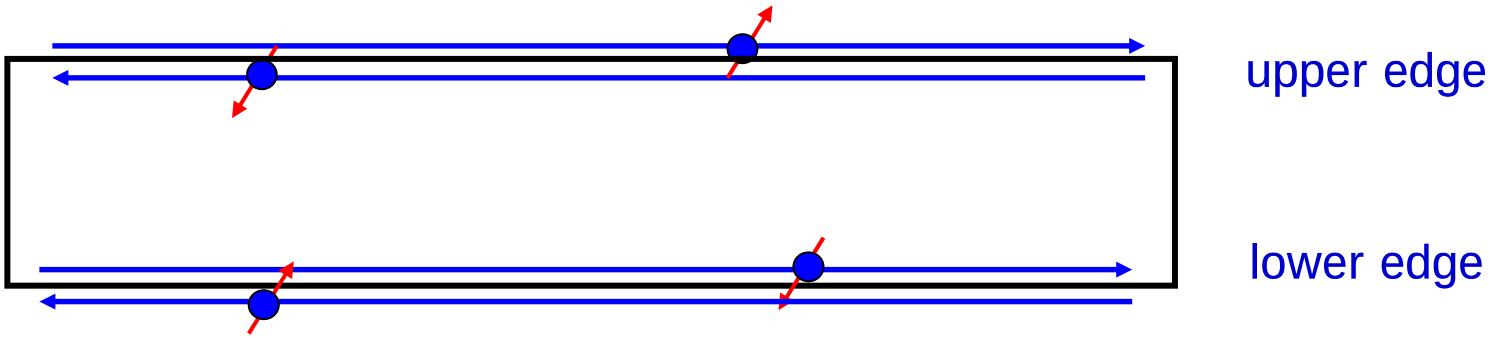
• **Stability of the gapless edge modes against impurity, disorder, magnetic impurity under strong interactions.**

C. Wu, B. A. Bernevig, and S. C. Zhang, Phys. Rev. Lett 96, 106401 (2006);

C. Xu and J. Moore, PRB, 45322 (2006).

# QSHE edges: Helical Luttinger liquids (HLL)

- Edge modes are characterized by helicity.
- Right-movers with spin up, and left-movers with spin down:
- *n*-component HLL: *n*-branches of time-reversal pairs ($T^2=-1$).




- HLL with an odd number of components are special.

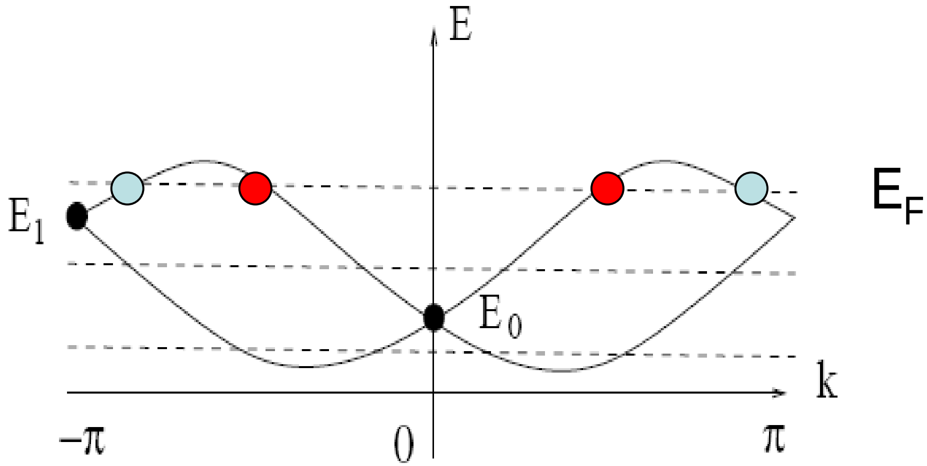
chiral Luttinger liquids in quantum Hall edges break TR symmetry;

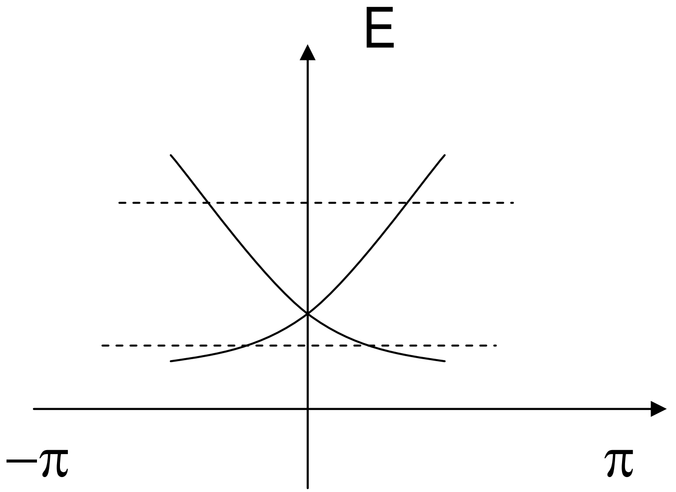
spinless non-chiral Luttinger liquids: $T^2=1$;

non-chiral spinful Luttinger liquids have an even number of branches of TR pairs.

# The no-go theorem for helical Luttinger liquids

• 1D HLL with an odd number of components can NOT be constructed in a purely 1D lattice system.




• Double degeneracy occurs at k=0 and $\pi$.

• Periodicity of the Brillouin zone.

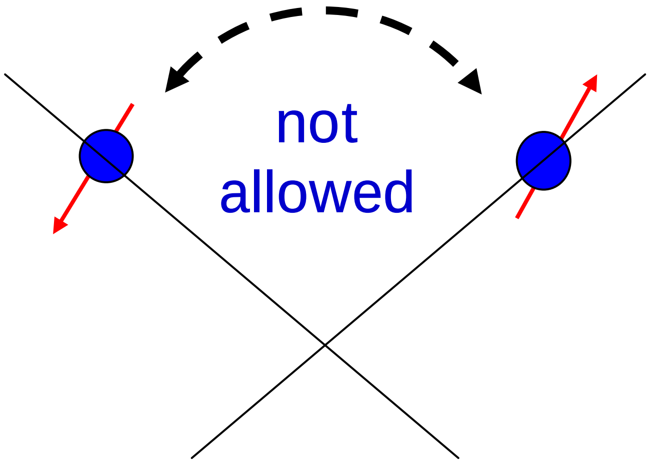
• HLL with an odd number of components can appear as the edge states of a 2-D system.

H. B. Nielsen et al., Nucl Phys. B 185, 20 (1981); C. Wu et al., Phys. Rev. Lett. **96**, 106401 (2006).

# Instability: the single-particle back-scattering

- The non-interacting Hamiltonian.

$$H_0 = v_f \int dx(\psi^+_{R\uparrow} i\partial_x \psi_{R\uparrow} - \psi^+_{L\downarrow} i\partial_x \psi_{L\downarrow})$$




- Kane and Mele : The non-interacting helical systems with an odd number of components remain gapless against disorder and impurity scatterings.

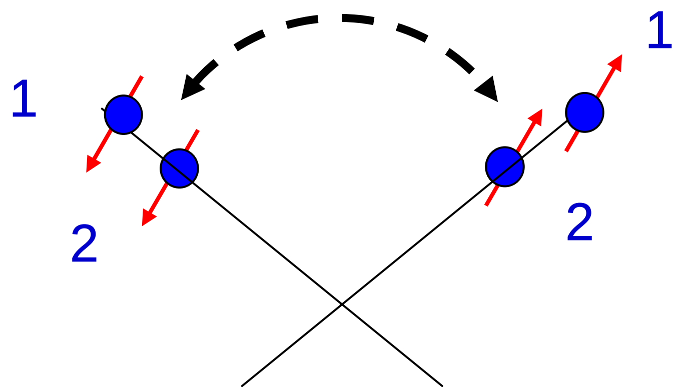
- Single particle backscattering term breaks TR symmetry ($T^2$=-1).

$$H_{bg} = \psi^+_{R\uparrow}\psi_{L\downarrow} + \psi^+_{L\downarrow}\psi_{R\uparrow} \qquad T^{-1} H_{bg} T = -H_{bg}$$

- However, with strong interactions, HLL can indeed open the gap from another mechanism.

# Two-particle correlated back-scattering

- TR symmetry allows two-particle correlated back-scattering.




$$G_2(t,0) = \left\langle G \right| \psi_{R\uparrow 1}(t)\psi_{R\uparrow 2}(t)\ \psi^{+}_{L\downarrow 2}(0)\psi^{+}_{L\downarrow 1}(0) \left| G \right\rangle_{connected} \neq 0$$

$$H_{um} = \sum_{\langle ij \rangle} s_x(i)s_x(j) - s_y(i)s_y(j),$$

$$\text{or} \sum_{\langle ij \rangle} s_x(i)s_y(j) + s_y(i)s_x(j)$$

- Microscopically, this Umklapp process can be generated from anisotropic spin-spin interactions.

- Effective Hamiltonian: $H_{um} = g_u \int dx e^{i4k_f} \psi^{+}_{R\uparrow}(x)\psi^{+}_{R\uparrow}(x+\varepsilon)\psi_{L\downarrow}(x+\varepsilon)\psi_{L\downarrow}(x) + h.c.$

- U(1) rotation symmetry → $Z_2$. $s_x \to -s_x, \quad s_y \to -s_y, \quad s_z \to s_z$

# Bosonization+Renormalization group

- Sine-Gordon theory if the Fermi wave vector is commensurate $k_f=\pi/2$.

$$\psi_{R\uparrow} \propto e^{i\sqrt{4\pi}\phi_{R\uparrow}},\ \psi_{L\downarrow} \propto e^{-i\sqrt{4\pi}\phi_{L\downarrow}};\ \phi(\theta)=\phi_{R\uparrow}\pm\phi_{L\downarrow}$$

$$H_0 = \int dx \frac{v}{2}\{\frac{1}{K}(\partial_x\phi)^2 + K(\partial_x\theta)^2\} + \frac{g_u}{2(\pi a)^2}\cos\sqrt{16\pi}\phi$$

- If K<1/2 (strong repulsive interaction), the gap $\Delta$ opens. Order parameters $2k_f$ SDW orders $N_x$ ($g_u<0$) or $N_y$ at ($g_u>0$) .

$$\frac{dg_u}{d\ln t} = (2-4K)g_u \qquad \Delta \approx a^{-1}(g_u)^{\frac{1}{4(1/2-K)}} \qquad \begin{aligned} N_x &\propto \cos\sqrt{4\pi}\phi, \\ N_y &\propto \sin\sqrt{4\pi}\phi \end{aligned}$$

- TR symmetry is spontaneously broken in the ground state.
- At $\Delta >> T > 0\mathrm{K}$, TR symmetry must be restored by thermal fluctuations and the gap remains.

# Random two-particle back-scattering

• Scattering amplitudes $g_u(x)e^{i\alpha(x)}$ are quenched Gaussian variables.

$$H_{\text{int}} = \int dx \frac{g_u(x)}{2(\pi a)^2} \cos(\sqrt{16\pi}\phi + \alpha(x))$$

$$\left\langle g_u(x)e^{i\alpha(x)} \; g_u(y)e^{-i\alpha(y)} \right\rangle = D\,\delta(x-y) \qquad \frac{dD}{d\ln t} = (3-8K)D$$

Giamarchi, *Quantum physics in one dimension,* oxford press (2004).

• If $K_c$<3/8, gap Δ opens. SDW order is spatially disordered but static in the time domain.

• TR symmetry is spontaneously broken.

• At small but finite temperatures, gap remains but TR is restored by thermal fluctuations.

# Single impurity scattering

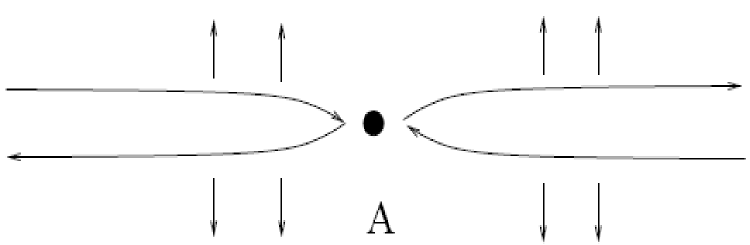
- Boundary Sine-Gordon equation.

$$H_{\text{int}} = \int dx \frac{g_u}{2(\pi a)^2} \delta(x) \cos(\sqrt{16\pi}\phi) \qquad \frac{dg_u}{d \ln t} = (1 - 4K) g_u$$

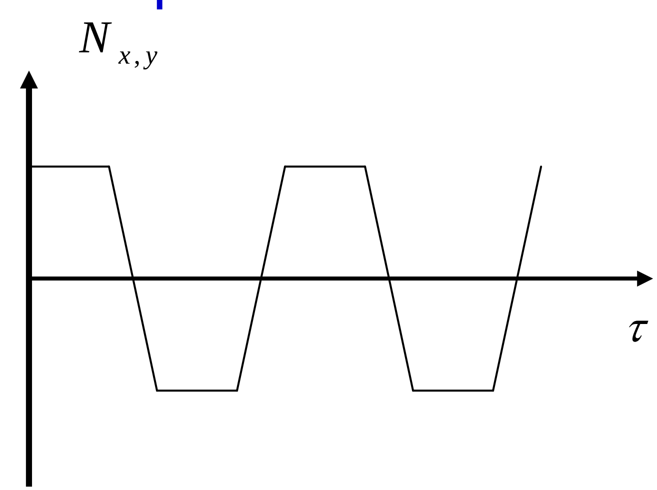
C. Kane and M. P. A. Fisher, PRB 46, 15233 (1992).

- If K<1/4, $g_u$ term is relevant. 1D line is divided into two segments.

- During each instanton process, half an electron tunnels, which is due to the backscattering of two particles.



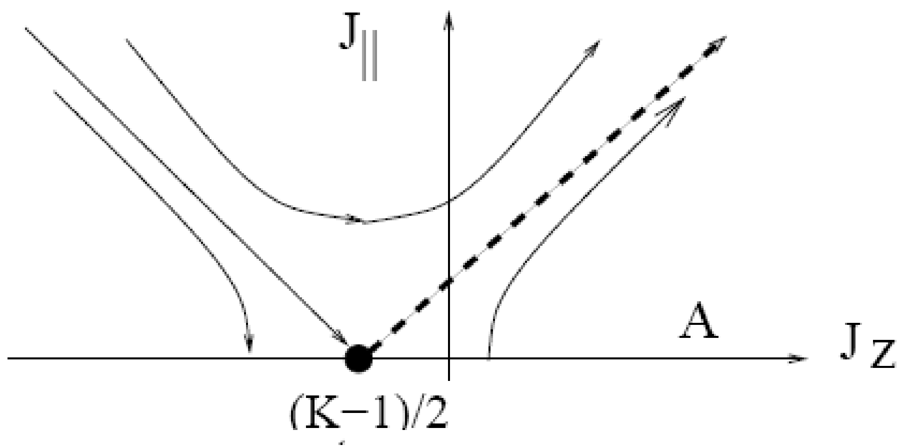

$$N_x \propto \cos\sqrt{4\pi}\phi, N_y \propto \sin\sqrt{4\pi}\phi$$



# Kondo problem: magnetic impurity scattering

$$H_K = \int dx\,\delta(x)\{\frac{J_{//}}{2}(\sigma_- \psi^+_{R\uparrow}\psi_{L\downarrow} + \sigma_+ \psi^+_{L\downarrow}\psi_{R\uparrow}) + J_z \sigma_z (\psi^+_{R\uparrow}\psi_{R\uparrow} - \psi^+_{L\downarrow}\psi_{L\downarrow})\}$$

- Poor man RG: critical coupling $J_z$ is shifted by interactions.
- If K<1 (repulsive interaction), the Kondo singlet can form with ferromagnetic couplings.




$$\frac{dJ_z}{d\ln t} = 2J_{//}^2,$$

$$\frac{dJ_{//}}{d\ln t} = (1 - K + 2J_z)J_{//}^2$$

# Summary

- Helical Luttinger liquid (HLL) as edge states of QSHE systems.

- No-go theorem: HLL with odd number of components can not be constructed in a purely 1D lattice system.

- Instability problem: Two-particle correlated back-scattering is allowed by TR symmetry, and becomes relevant at:

  $K_c<1/2$ for Umklapp scattering at commensurate fillings.

  $K_c<3/8$ for random disorder scattering.

  $K_c<1/4$ for a single impurity scattering.

- Critical Kondo coupling $J_z$ is shifted by interaction effects.