Classification of topological insulators and superconductors

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question

How many different topoloigcal insulators and superconductors are there in nature ?

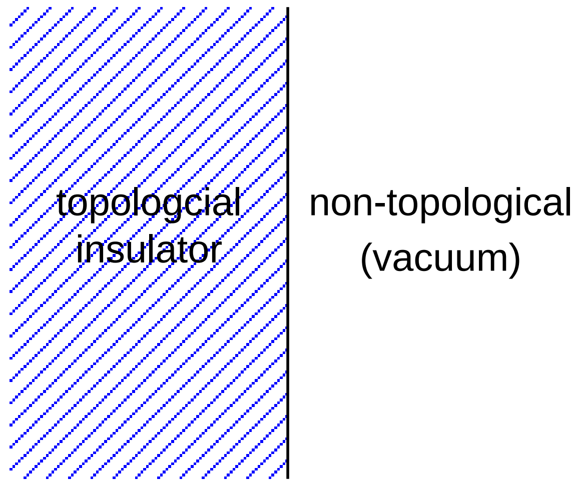
How many different topoloigcal insulators and superconductors are there in nature ?

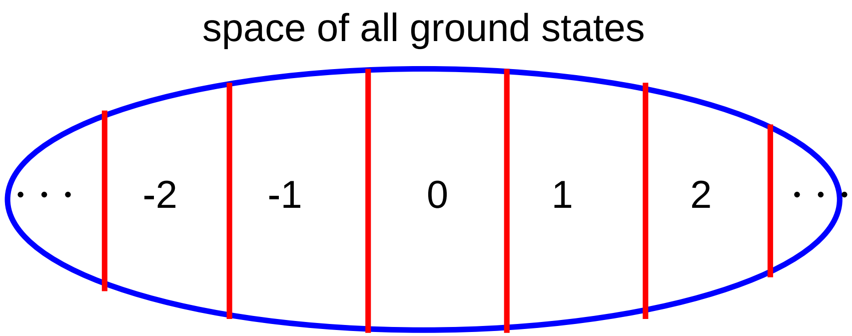
topological:

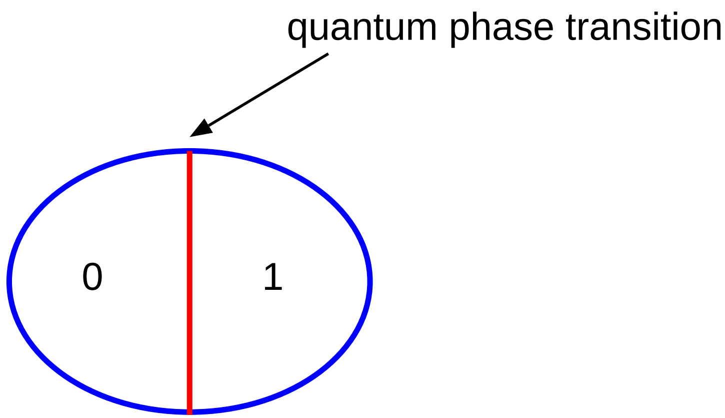
- support stable gapless modes at boundaries, possibly in the presence of general discrete symmetries

- states with and without boundary modes are not adiabatically connected

- may be characterized by a bulk topological invariant of some sort







topological insulators; examples

(i) IQHE in 2D, strong T breaking by B

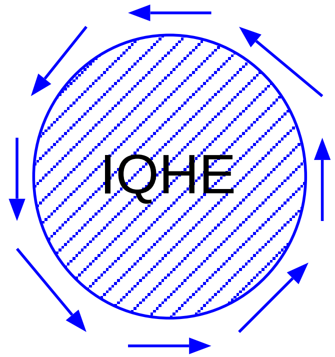
a) quantized Hall conductance

$$\sigma_{xy} \in \mathbf{Z} \times \frac{e^2}{h}$$

TKNN (82)
Laughlin (81)

b) stable edge states

Halperin (82)



(ii) Z2 topological insulator (QSHE) in 2D

(iii) Z2 topological insulator in 3D

TRI

$$i\sigma_y \mathcal{H}^T (-i\sigma_y) = \mathcal{H}$$

- characterized by Z2 topological number $\Delta = 0, 1$

- stable edge/surface states

classification of discrete symmetries

-natural framework: random matrix theory (RMT)

Wigner-Dyson

Zirnbauer (96), Altland &Zirnbauer (97)

two types of anti-unitary symmetries

Time-Reversal Symmetry (TRS)

$$\mathcal{T}\mathcal{H}^*\mathcal{T}^{-1} = \mathcal{H}$$

TRS = { 0 no TRS; +1 TRS with $\mathcal{T}^T = +\mathcal{T}$ (integer spin particle); -1 TRS with $\mathcal{T}^T = -\mathcal{T}$ (half-odd integer spin particle) }

Particle-Hole Symmetry (PHS)

$$C\mathcal{H}^T C^{-1} = -\mathcal{H}$$

PHS = { 0 no PHS; +1 PHS with $C^T = +C$; -1 PHS with $C^T = -C$ }

PHS + TRS = chiral symmetry

$$\left.\begin{array}{l} T\mathcal{H}^*T^{-1} = \mathcal{H} \\ C\mathcal{H}^*C^{-1} = -\mathcal{H} \end{array}\right\} \longrightarrow TC\mathcal{H}(TC)^{-1} = -\mathcal{H}$$

classification of discrete symmetries

-natural framework: random matrix theory (RMT)

Wigner-Dyson   Zirnbauer (96), Altland &Zirnbauer (97)

		TRS	PHS	SLS	description	RM ensembles
Wigner-Dyson (standard)	A	0	0	0	unitary	$U(N)$
	AI	+1	0	0	orthogonal	$U(N)/O(N)$
	AII	-1	0	0	symplectic (spin-orbit)	$U(2N)/Sp(N)$
chiral (sublattice)	AIII	0	0	1	chiral unitary	$U(2N)/U(N) \times U(N)$
	BDI	+1	+1	1	chiral orthogonal	$O(2N)/O(2N) \times O(2N)$
	CII	-1	-1	1	chiral symplectic	$Sp(4N)/Sp(2N) \times Sp(2N)$
BdG	D	0	+1	0	singlet/triplet SC	$O(N)$
	C	0	-1	0	singlet SC	$Sp(N)$
	DIII	-1	+1	1	singlet/triplet SC with TRS	$O(2N)/U(N)$
	CI	+1	-1	1	singlet SC with TRS	$Sp(N)/U(N)$

classification of discrete symmetries

-natural framework: random matrix theory (RMT)

Wigner-Dyson   Zirnbauer (96), Altland &Zirnbauer (97)

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	CII	-1	-1	1	chiral symplectic
BdG	D	0	+1	0	singlet/triplet SC
	C	0	-1	0	singlet SC
	DIII	-1	+1	1	singlet/triplet SC with TRS
	CI	+1	-1	1	singlet SC with TRS

-IQHE is a topological insulator in unitary class (A).

-Z2 toplological insulator is a topological insulator in symplectic class (AII).

-Is there a topological insulator in other symmetry classes ?

BdG symmetry classes

- S^z non-conserving SC

$$H = \frac{1}{2}\left(\mathbf{c}_{\uparrow}^{\dagger}, \mathbf{c}_{\downarrow}^{\dagger}, \mathbf{c}_{\uparrow}, \mathbf{c}_{\downarrow}\right) \mathcal{H} \begin{pmatrix} \mathbf{c}_{\uparrow} \\ \mathbf{c}_{\downarrow} \\ \mathbf{c}_{\uparrow}^{\dagger} \\ \mathbf{c}_{\downarrow}^{\dagger} \end{pmatrix} \qquad \mathcal{H} = \begin{pmatrix} \xi & \Delta \\ -\Delta^* & -\xi^T \end{pmatrix} \qquad \xi = \xi^{\dagger}, \quad \Delta = -\Delta^T$$

	TR	SU(2)		examples in 2D
D	×	×	$\tau_x \mathcal{H}^T \tau_x = -\mathcal{H}$	spinless chiral p-wave
DIII	○	×	$\tau_x \mathcal{H}^T \tau_x = -\mathcal{H},\ \sigma_y \mathcal{H}^T \sigma_y = \mathcal{H}$	p-wave

- S^z conserving SC

$$H = (\mathbf{c}_{\uparrow}^{\dagger}, \mathbf{c}_{\downarrow})\mathcal{H} \begin{pmatrix} \mathbf{c}_{\uparrow} \\ \mathbf{c}_{\downarrow}^{\dagger} \end{pmatrix} \qquad \mathcal{H} = \begin{pmatrix} \xi_{\uparrow} & \Delta \\ \Delta^{\dagger} & -\xi_{\downarrow}^T \end{pmatrix} \qquad \xi_{\sigma} = \xi_{\sigma}^{\dagger}$$

	TR	SU(2)		examples in 2D
A	×	△	no constraint	spinfull chiral p-wave
AIII	○	△	$\tau_y \mathcal{H} \tau_y = -\mathcal{H}$	p-wave
C	×	○	$\tau_y \mathcal{H}^T \tau_y = -\mathcal{H}$	d+id -wave
CI	○	○	$\tau_y \mathcal{H}^T \tau_y = -\mathcal{H},\ \mathcal{H}^* = \mathcal{H}$	d-wave, s-wave

sublattice symmetry classes

$$H = (\mathbf{c}_A^\dagger, \mathbf{c}_B^\dagger)\mathcal{H}\begin{pmatrix} \mathbf{c}_A \\ \mathbf{c}_B \end{pmatrix} \qquad \mathcal{H} = \begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} \qquad \gamma\mathcal{H} = -\mathcal{H}\gamma \qquad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

	TR	SU(2)		examples	
AIII	×	× ○	$\tau_y \mathcal{H} \tau_y = -\mathcal{H}$	random flux model	
BDI	○	○	$\tau_y \mathcal{H} \tau_y = -\mathcal{H},\ \mathcal{H}^* = \mathcal{H}$	random hopping model	Dyson (53)
CII	○	×	$\tau_y \mathcal{H} \tau_y = -\mathcal{H},\ \sigma_y \mathcal{H}^T \sigma_y = \mathcal{H}$		Gade (93)

- Classes CI and DIII have an off-diagonal form ! (will be important later)

PHS + TRS = chiral (sublattice) symmetry

$$\left.\begin{aligned} T\mathcal{H}^T T^{-1} &= \mathcal{H} \\ C\mathcal{H}^T C^{-1} &= -\mathcal{H} \end{aligned}\right] \longrightarrow \quad TC\mathcal{H}(TC)^{-1} = -\mathcal{H}$$

classification of 3D topological insulators

Schnyder, SR, Furusaki, Ludwig (2008)

RESULT:

-3D topological insulators for 5 out of 10 symmetry classes

AIII, DIII, CI : top. insulators labeled by an integer
AII, CII: top. insulators of Z2 type

		TRS	PHS	SLS	description
Wigner-Dyson (standard)	A	0	0	0	unitary
	AI	+1	0	0	orthogonal
	AII	-1	0	0	symplectic (spin-orbit)
chiral (sublattice)	AIII	0	0	1	chiral unitary
	BDI	+1	+1	1	chiral orthogonal
	CII	-1	-1	1	chiral symplectic
BdG	D	0	+1	0	singlet/triplet SC
	C	0	-1	0	singlet SC
	DIII	-1	+1	1	singlet/triplet SC with TRS
	CI	+1	-1	1	singlet SC with TRS

classification of 3D topological insulators

Schnyder, SR, Furusaki, Ludwig (2008)

underlying strategy

- discover a topological invariant

integer topological invairant for 3 out of 5 classes

$$\nu = \int_{\mathrm{Bz}} \frac{d^3 k}{24\pi^2} \epsilon^{\mu\nu\rho} \mathrm{tr}\left[(q^{-1}\partial_\mu q)(q^{-1}\partial_\nu q)(q^{-1}\partial_\rho q) \right]$$

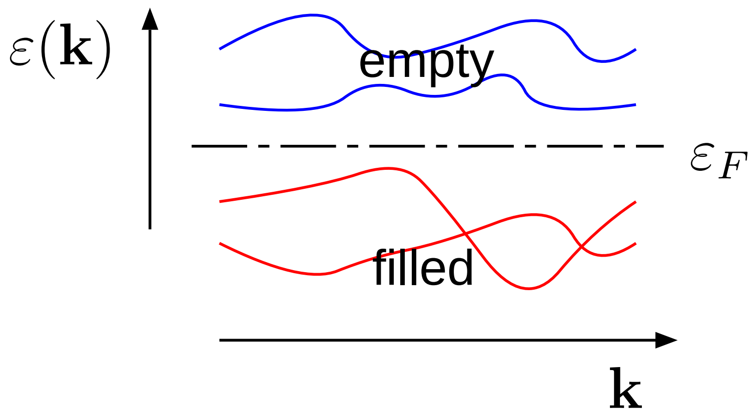
q : BZ $\longrightarrow U(m)$ spectral projector

- bulk-boundary correspondence

absence of Anderson localization at boundaries

topological distinction of ground states

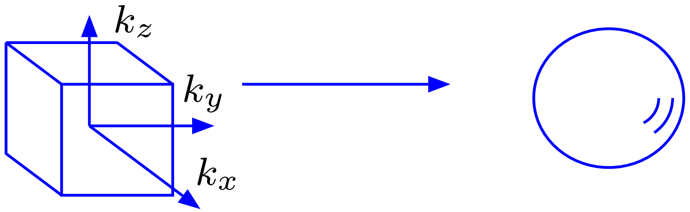
projector: $$Q(k) = 2 \sum_{a \in \text{filled}} |u_a(k)\rangle\langle u_a(k)| - 1$$



$$Q^2 = 1, \; Q^\dagger = Q, \; \mathrm{tr}\, Q = m - n$$

filled (→ m) empty (→ n)

Q : BZ ⟶ $U(m+n)/U(m) \times U(n)$



quantum ground state = map from Bz onto Grassmannian

$\pi_2[U(m+n)/U(m) \times U(n)] = \mathbf{Z}$ ⟶ IQHE in 2D

$\pi_3[U(m+n)/U(m) \times U(n)] = 0$

⟶ no top. insulator in 3D without constraint (Class A)

(for large enough m,n)

topological distinction of ground states

-projectors in classes AIII

chiral symmetry $\Gamma \mathcal{H} \Gamma = -\mathcal{H}$ $\longrightarrow$ $Q(k) = \begin{pmatrix} 0 & q(k) \\ q^\dagger(k) & 0 \end{pmatrix}$

q : BZ $\longrightarrow U(m)$

$\pi_3[U(m)] = \mathbf{Z}$ $\longrightarrow$ topological insulators labeled by an integer

$$\nu = \int_{\mathrm{Bz}} \frac{1}{24\pi^2} \mathrm{tr}\left[(q^{-1} dq)^3\right]$$

-discrete symmetries limit possible values of nu

$q^T(-k) = -q(k)$ DIII

$q^T(-k) = q(k)$ CI

$q^*(-k) = q(k)$ BDI

$i\sigma_y q^*(-k)(-i\sigma_y) = -q(k)$ CII

AIII & DIII $\nu \in \mathbf{Z}$

CI $\nu \in 2\mathbf{Z}$

CII & BDI $\nu = 0$

Z2 insulators in CII (later)

Anderson delocalization at boundaries

⟷ topological bulk

		TRS	PHS	SLS	fermionic replica NLsM	
Wigner-Dyson (standard)	A	0	0	0	$U(2N)/U(N)\times U(N)$	Pruisken
	AI	+1	0	0	$Sp(4N)/Sp(2N)\times Sp(2N)$	
	AII	-1	0	0	$O(2N)/O(2N)\times O(2N)$	$\mathbf{Z}_2$
chiral (sublattice)	AIII	0	0	1	$U(N)$	WZW
	BDI	+1	+1	1	$U(2N)/Sp(N)$	
	CII	-1	-1	1	$U(N)/O(N)$	$\mathbf{Z}_2$
BdG	D	0	+1	0	$O(2N)/U(N)$	Pruisken
	C	0	-1	0	$Sp(N)/U(N)$	Pruisken
	DIII	-1	+1	1	$O(N)$	WZW
	CI	+1	-1	1	$Sp(N)$	WZW

newly derived !

- Bernard-Le Clair: 13-fold symmetry classifcation of 2d Dirac fermions

- AIII, CI, DIII; exact results

- "abnormal terms" in NLsM

WZW type
$$Z=\int \mathcal{D}[g]e^{2\pi i\nu\Gamma_{\rm WZW}}e^{-S[g]} \qquad \Gamma_{\rm WZW}=\frac{1}{24\pi^2}\int_{\mathcal{M}^3}\mathrm{tr}\left[(g^{-1}dg)^3\right]$$

Z2 type
$$Z=\int \mathcal{D}[Q](-1)^{N[Q]}e^{-S[Q]}$$

SR, Mudry, Obuse Furusaki (07)

characterization at boundaries

-classification of 2D Dirac Hamiltonians

Bernard-LeClair (2001)

$$\mathcal{H} = \begin{pmatrix} V_+ + V_- & -i\bar{\partial} + A_+ \\ +i\partial + A_- & V_+ - V_- \end{pmatrix}$$

13 classes (not 10 !)

AIII, CI, DIII has an extra class.

		TR	SU(2)	description
Wigner-Dyson (standard)	A	×	○ ×	unitary
	AI	○	○	orthogonal
	AII	○	×	symplectic (spin-orbit)
chiral (sublattice)	AIII	×	○ ×	chiral unitary
	AIII	×	○ ×	chiral unitary extra
	BDI	○	○	chiral orthogonal
	CII	○	×	chiral symplectic
BdG	C	×	○	singlet SC
	D	×	×	singlet/triplet SC
	CI	○	○	singlet SC
	CI	○	○	singlet SC extra
	DIII	○	×	singlet/triplet SC
	DIII	○	×	singlet/triplet SC extra

even/odd effect

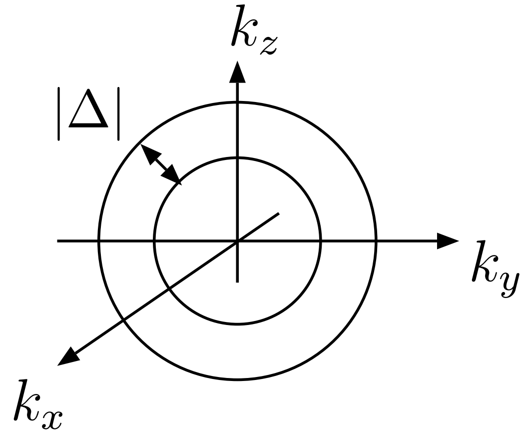
always gapless

3He is a 3D topological insulator

- Class DIII top. insulator: B 3He

$$\mathcal{H} = \begin{pmatrix} \xi & \Delta \\ -\Delta^* & -\xi^T \end{pmatrix}$$

$$\xi_{\mathbf{k}} = \frac{k^2}{2m} - \mu \qquad \Delta_{\mathbf{k}} = |\Delta| i\sigma_y \mathbf{k}\cdot\sigma$$



isotropic gap

strong pairing | weak pairing

μ

$\nu = 0$ | 0 | $\nu = 1$

Z2 classification:

Roy (2008)

Qi-Hughes-Raghu-Zhang (2008)

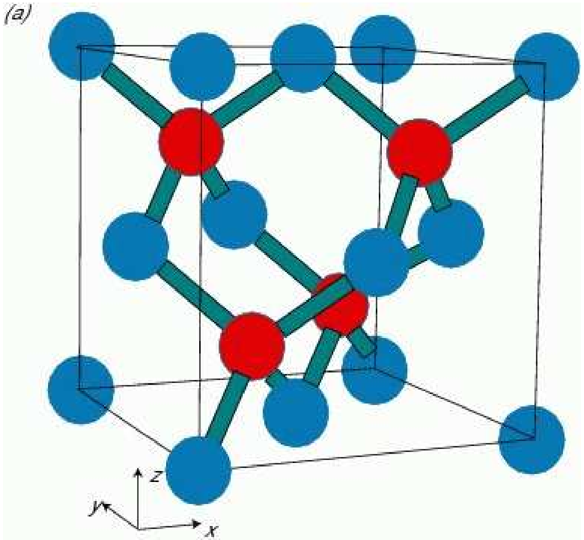
-stable surface Majorana fermion state

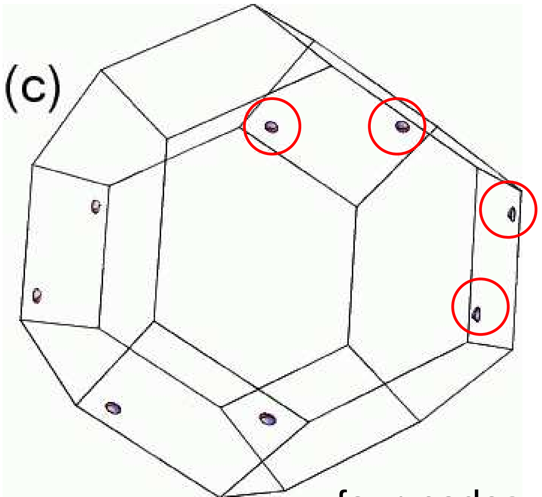
Salomma and Volovik (1988)

topological singlet superconductor in 3D

- class CI top. insulator: singlet BCS pairing model on the diamond lattice

SU(2) symmetric

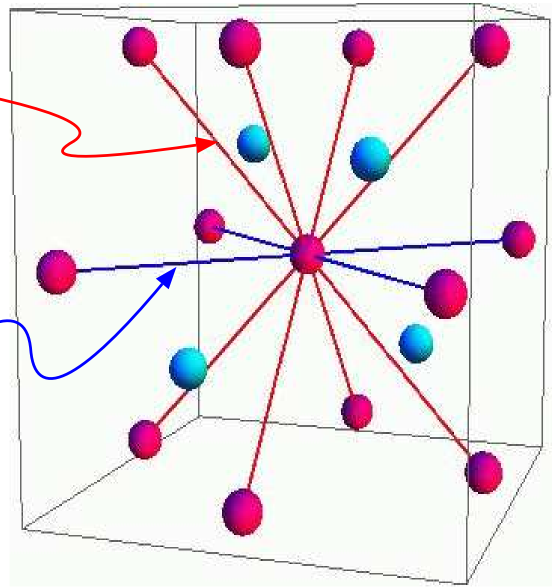




four-nodes, two-fold degenerate
= two 8-component Dirac

$d_{3z^2-r^2}$-wave like (?) pairing

$\Delta > 0$



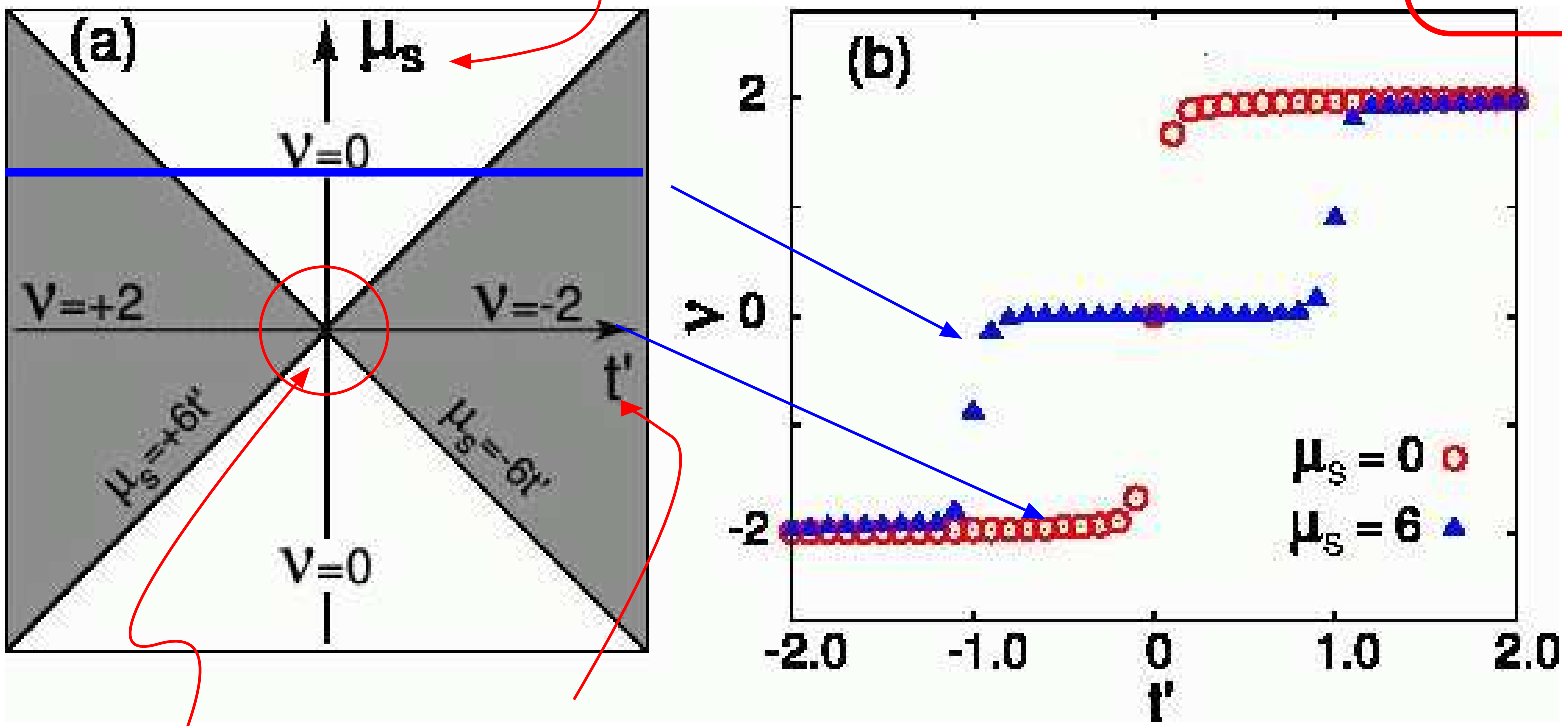
$\Delta' < 0$

t nn: hopping
Delta: nn d-wave pairing
t': nnn hopping
mu_s: staggered chemical potential

topological singlet superconductor in 3D

staggered chemical potential

$\nu = \pm 2$



nnn hopping

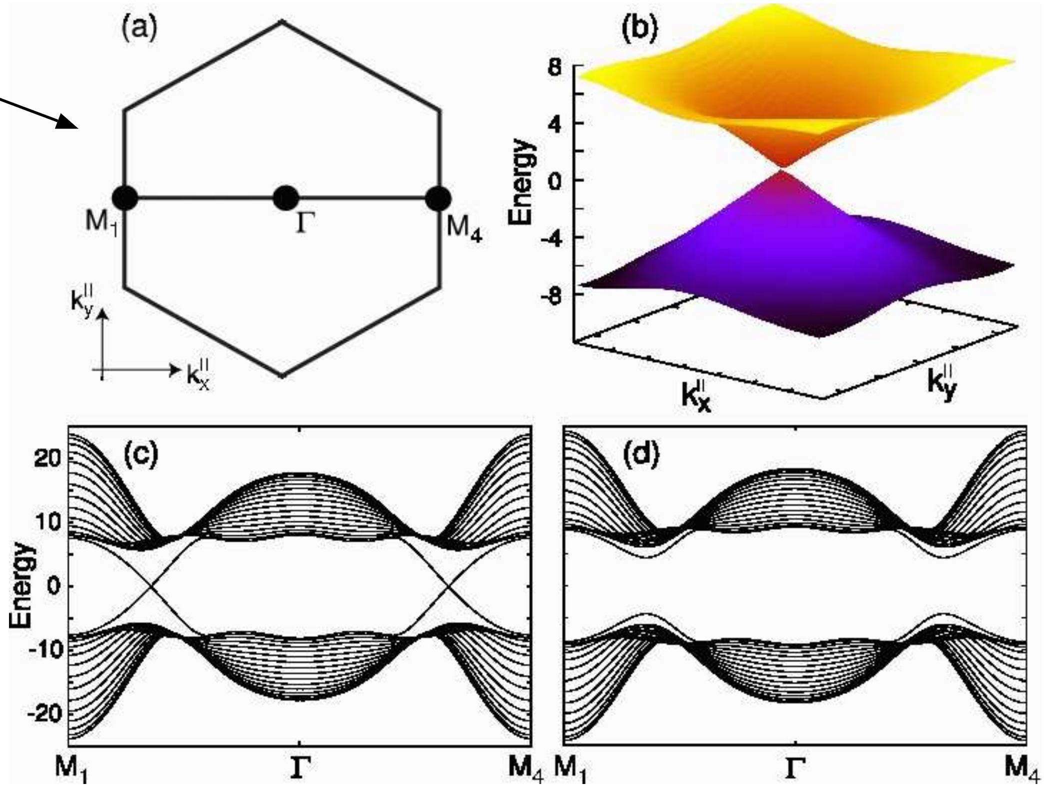
nnn hopping

t and Delta only

$$\nu = \int_{\mathrm{Bz}} \frac{d^3k}{24\pi^2} \epsilon^{\mu\nu\rho} \mathrm{tr}\left[(q^{-1}\partial_\mu q)(q^{-1}\partial_\nu q)(q^{-1}\partial_\rho q)\right]$$

surface of 3d top. singlet SC = "1/2 of cuprate"

surface Bz



-- stable surf. Dirac fermions

$$\sigma^{\text{spin}} = \frac{1}{\pi} \times 2 \times N \times \frac{s^2}{h}$$

(irrespective of disorder strength)

-- surface Dirac fermion + disorder

→ exactly solvable

Tsvelik (1995)

-- T-breaking -> half spin quantum Hall effect ("1/2 of d+id SC")

summary

- 3D topological insulators for 5 out of 10 symmetry classes.

  AIII, DIII, CI : top. insulators labeled by an integer

  AII, CII: top. insulators of Z2 type

- Topological insulator /Anderson delocalization correspondence

  surface of top. insulator is always conducting.

- The same strategy is applicable to other dimensions.

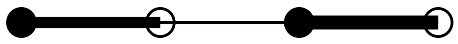
- Transport experiments on Bismuth-Antimony ?

  perfectly conducting because of Z2 topological term

- Topological field theory ?

$$S = \frac{\theta}{32\pi^2} \int d^4x \, \epsilon^{\mu\nu\rho\lambda} \mathrm{tr} \, [F_{\mu\nu} F_{\rho\lambda}] \qquad A_\mu \in \mathrm{SU}(2)$$

summary


polyacetylene

QSHE

IQHE

Z2 top. ins

		TRS	PHS	SLS	description	d=1	d=2	d=3
Wigner-Dyson (standard)	A	0	0	0	unitary	0	$\mathbb{Z}$	0
	AI	+1	0	0	orthogonal	0	0	0
	AII	-1	0	0	symplectic (spin-orbit)	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
chiral (sublattice)	AIII	0	0	1	chiral unitary	$\mathbb{Z}$	0	$\mathbb{Z}$
	BDI	+1	+1	1	chiral orthogonal	$\mathbb{Z}$	0	0
	CII	-1	-1	1	chiral symplectic	$\mathbb{Z}$	0	$\mathbb{Z}_2$
BdG	D	0	+1	0	singlet/triplet SC	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	C	0	-1	0	singlet SC	0	$\mathbb{Z}$	0
	DIII	-1	+1	1	singlet/triplet SC with TRS	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	CI	+1	-1	1	singlet SC with TRS	0	0	$\mathbb{Z}$

chiral p-wave SC

d+id wave SC

3He B

- 3D weak topological insulators are also possible in classes A, AII, D, C, CI

topological field theory description

- generating function for single particle Green's function

$$Z = \int \mathcal{D}[\psi^\dagger, \psi] e^{-\int d^3x \mathcal{L}} \qquad \mathcal{L} = \psi^\dagger i(\mathcal{H} - i\eta)\psi$$

(3+0) dim field theory

- introduce external gauge fields

$$\mathcal{L} = \bar{\psi}(\partial_\mu \gamma_\mu - i a_\mu \gamma_\mu - i b_\mu \gamma_0 \gamma_\mu + m\gamma_5)\psi$$

- integrate over fermions

$$e^{-S_{\text{eff}}[a_\mu, b_\mu]} = \int \mathcal{D}[\bar{\psi}, \psi] e^{-S[a_\mu, b_\mu, \bar{\psi}, \psi]}$$

$$\longrightarrow \quad S_{\text{eff}} = \nu \left(I[A^+] - I[A^-] \right) \qquad A^\pm_\mu = a_\mu \pm b_\mu$$

$$I[A] = \frac{-i}{4\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} \left[A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda \right]$$

non Abelian doubled Chern-Simons