High-Order Computations in Numerical Stochastic Perturbation Theory: An Intriguing Opportunity for Probing Resurgence

F. Di Renzo
University of Parma and INFN

UNIVERSITÀ DI PARMA

INFN Istituto Nazionale di Fisica Nucleare

Resurgence in Gauge and String Theory

KITP UCSB, Nov 02, 2017

An invitation (original motivations…)

Perturbation Theory (PT) is nothing less than ubiquitous in Field Theory. In principle the lattice is a regulator among the others ... in practice it is a dreadful one so that when it comes to compute something in Lattice Perturbation Theory (LPT) you will probably start to get nervous ...

In particular for LGT:
lot of vertices (not given once and for all)
Sums and/or integrals ... a lot of trigonometrics ...
A variety of actions (both for glue and for quarks)

and as an extra bonus ... often bad convergence properties

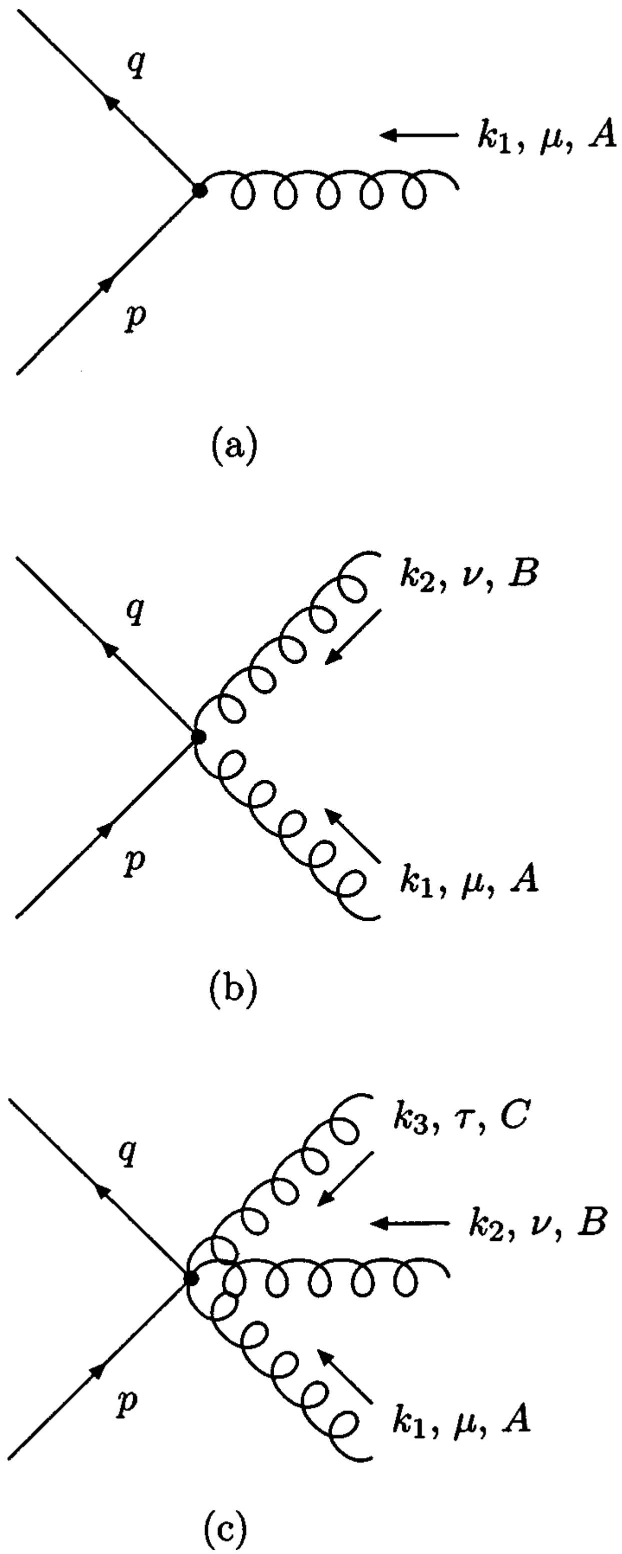


FIG. 1. Momentum assignments for the quark-antiquark-gluon vertices.

$$V^{A}_{1\mu}(p,q) = -gT^{A}\left[i\gamma_{\mu}\cos\left(\frac{p_{\mu}+q_{\mu}}{2}\right) + r\sin\left(\frac{p_{\mu}+q_{\mu}}{2}\right)\right]$$

$$V^{A}_{c1\mu}(p,q) = -gT^{A}c_{\mathrm{SW}}\frac{r}{2}\sum_{\nu}\sigma_{\mu\nu} \times\cos\left(\frac{p_{\mu}-q_{\mu}}{2}\right)\sin(p_{\nu}-q_{\nu})$$

$$V^{AB}_{2\mu\nu}(p,q) = \frac{a}{2}g^{2}\frac{1}{2}\{T^{A},T^{B}\}\delta_{\mu\nu} \times\left[i\gamma_{\mu}\sin\left(\frac{p_{\mu}+q_{\mu}}{2}\right) - r\cos\left(\frac{p_{\mu}+q_{\mu}}{2}\right)\right]$$

$$V^{AB}_{c2\mu\nu}(p,q,k_{1},k_{2}) = -\frac{a}{2}g^{2}if_{ABC}T^{C}c_{\mathrm{SW}}\frac{r}{4}\left\{\sigma_{\mu\nu}\left[4\cos\left(\frac{k_{1\nu}}{2}\right)\cos\left(\frac{k_{2\mu}}{2}\right)\cos\left(\frac{q_{\mu}-p_{\mu}}{2}\right) \times\cos\left(\frac{q_{\nu}-p_{\nu}}{2}\right) - 2\cos\left(\frac{k_{1\mu}}{2}\right)\cos\left(\frac{k_{2\nu}}{2}\right)\right] + \delta_{\mu\nu}\sum_{\rho}\sigma_{\mu\rho}\sin\left(\frac{q_{\mu}-p_{\mu}}{2}\right)[\sin(k_{2\rho})-\sin(k_{1\rho})]\right\}$$

$$V^{ABC}_{3\mu\nu\tau}(p,q) = \frac{a^{2}}{6}g^{3}\frac{1}{6}[T^{A}\{T^{B},T^{C}\}+T^{B}\{T^{C},T^{A}\} +T^{C}\{T^{A},T^{B}\}]\delta_{\mu\nu}\delta_{\mu\tau}\left[i\gamma_{\mu}\cos\left(\frac{p_{\mu}+q_{\mu}}{2}\right) +r\sin\left(\frac{p_{\mu}+q_{\mu}}{2}\right)\right],$$

$$V^{ABC}_{c3\mu\nu\tau}(p,q,k_{1},k_{2},k_{3}) = -3ig^{3}\frac{a^{2}}{6}c_{\mathrm{SW}}r\left\{T^{A}T^{B}T^{C}\delta_{\mu\nu}\delta_{\mu\tau}\sum_{\rho} i\sigma_{\mu\rho}\left[-\frac{1}{6}\cos\left(\frac{q_{\mu}-p_{\mu}}{2}\right)\sin(q_{\rho}-p_{\rho}) +\cos\left(\frac{q_{\mu}-p_{\mu}}{2}\right)\cos\left(\frac{q_{\rho}-p_{\rho}}{2}\right)\cos\left(\frac{k_{3\rho}-k_{1\rho}}{2}\right)\sin\left(\frac{k_{2\rho}}{2}\right)\right] - \frac{1}{2}[T^{A}T^{B}T^{C} +T^{C}T^{B}T^{A}]i\sigma_{\mu\nu}\left[\delta_{\nu\tau}2\cos\left(\frac{q_{\mu}-p_{\mu}}{2}\right)\cos\left(\frac{q_{\nu}-p_{\nu}}{2}\right)\cos\left(\frac{k_{3\mu}+k_{2\mu}}{2}\right)\sin\left(\frac{k_{1\nu}}{2}\right) +\delta_{\nu\tau}\sin\left(\frac{k_{3\nu}+k_{2\nu}}{2}\right)\cos\left(\frac{k_{1\mu}}{2}+k_{2\mu}\right) + \delta_{\mu\tau}\sin\left(\frac{k_{1\mu}+2k_{2\mu}+k_{3\mu}}{2}\right)\cos\left(\frac{q_{\nu}-p_{\nu}}{2}\right)\cos\left(\frac{k_{3\nu}-k_{1\nu}}{2}\right)\right.$$

Despite this ...

ELSEVIER

Nuclear Physics B 457 (1995) 202–216

NUCLEAR
PHYSICS B

Renormalons from eight-loop expansion of the gluon condensate in lattice gauge theory ★

F. Di Renzo [a], E. Onofri [a], G. Marchesini [b]

[a] *Dipartimento di Fisica, Università di Parma and INFN, Gruppo Collegato di Parma, Italy*
[b] *Dipartimento di Fisica, Università di Milano and INFN, Sezione di Milano, Italy*

Received 3 April 1995; revised 31 July 1995; accepted 5 October 1995

PRL **108,** 242002 (2012) PHYSICAL REVIEW LETTERS week ending 15 JUNE 2012

Compelling Evidence of Renormalons in QCD from High Order Perturbative Expansions

Clemens Bauer,[1] Gunnar S. Bali,[1] and Antonio Pineda[2]

[1] *Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany*
[2] *Grup de Física Teòrica, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Barcelona, Spain*

(Received 16 November 2011; published 12 June 2012)

We compute the static self-energy of SU(3) gauge theory in four spacetime dimensions to order α^{20} in the strong coupling constant α. We employ lattice regularization to enable a numerical simulation within the framework of stochastic perturbation theory. We find perfect agreement with the factorial growth of high order coefficients predicted by the conjectured renormalon picture based on the operator product expansion.

DOI: 10.1103/PhysRevLett.108.242002 PACS numbers: 12.38.Cy, 11.10.Jj, 11.15.Bt, 12.38.Bx

Motivation: (leading order…) RESURGENCE!

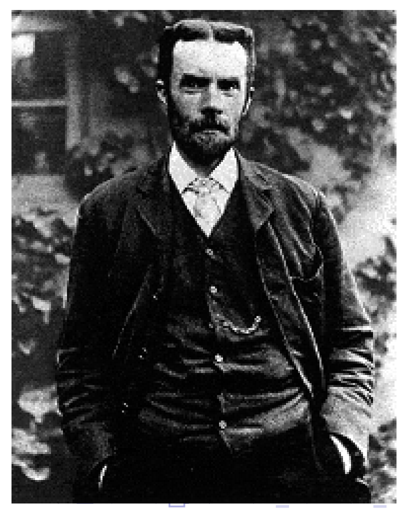
From Gerald Dunne's lectures at the Parma School 2016
(Decoding the path integral: resurgence, Lefschetz thimbles, non-perturbative physics)

Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever … That most of these things [summation of divergent series] are correct, in spite of that, is extraordinarily surprising. I am trying to find a reason for this; it is an exceedingly interesting question.

N. Abel, 1802 – 1829

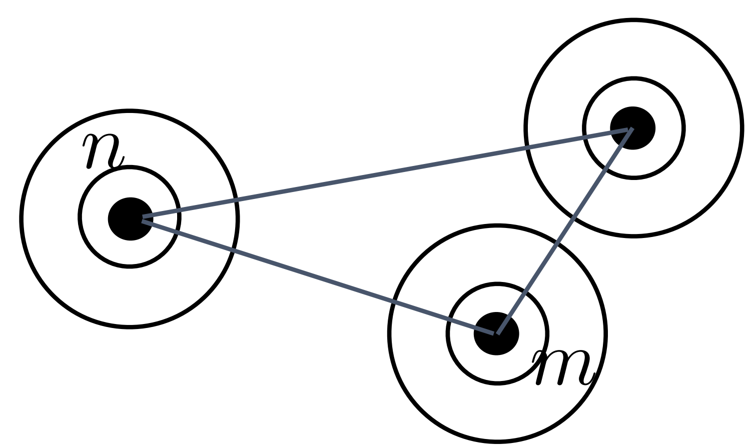
The series is divergent; therefore we may be able to do something with it

O. Heaviside, 1850 – 1925



resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



Motivation: (leading order…) RESURGENCE!

From Mitat Unsal's presentation at LATTICE2015

Simpler question: Can we make sense of the semi-classical expansion of QFT?

Argyres, MÜ, Dunne, MÜ, 2012

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)}\,(\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n}\, e^{-n\,A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)}\,(\lambda\hbar)^k$$

pert. th. n-instanton factor pert. th. around n-instanton

All series appearing above are asymptotic, i.e., divergent as $c_{(0,k)} \sim k!$. The combined object is called trans-series following resurgence terminology.

resurgence: fluctuations about the instanton/anti-instanton saddle are determined by those about the vacuum saddle.

Motivation: (leading order…) RESURGENCE!

QM:
low order/low order relations!

$$-\frac{\hbar^2}{2}\frac{d^2}{dx^2}\psi + V(x)\psi = u\,\psi$$

$$u_\pm(\hbar, N) = u_{\rm pert}(\hbar, N) \pm \sqrt{\frac{2}{\pi}}\frac{1}{N!}\left(\frac{2^{7/2}}{\hbar}\right)^{N+\frac{1}{2}} \exp\left[-\frac{2\sqrt{2}}{\hbar}\right]\mathcal{P}_{\rm inst}(\hbar, N) + \dots$$

$$u_{\rm pert}(\hbar, N) = \sum_{n=0}^{\infty}\hbar^n u_n(N)\,, \qquad \mathcal{P}_{\rm inst}(\hbar, N) = \sum_{n=0}^{\infty}\hbar^n p_n(N)$$

$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial u_{\rm pert}(\hbar, N)}{\partial N}\exp\left[\frac{S_{\mathcal{I}}}{\omega_c}\int_0^{\hbar}\frac{d\hbar}{\hbar^3}\left(\frac{\partial u_{\rm pert}(\hbar, N)}{\partial N} - \hbar\omega_c + \frac{\hbar^2\omega_c\left(N+\frac{1}{2}\right)}{S_{\mathcal{I}}}\right)\right]$$

PROBLEM!

Computations sometimes require (e.g. around instantons) heroic efforts!

Agenda

- Basics of Stochastic Quantization and Stochastic Perturbation Theory
- From Stochastic Perturbation Theory to NSPT (coming back to the QM problem)
- A few different frameworks for NSPT (i.e. a few handles to possibly improve it)

- NSPT around (euclidean QM) instantons!

- Conclusions

Basics of Stochastic Quantization and Stochastic Perturbation Theory

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You start with a field theory you want to solve

$$\langle O[\phi] \rangle = \frac{\int D\phi \ O[\phi] \ e^{-S[\phi]}}{\int D\phi \ e^{-S[\phi]}}$$

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Parisi-Wu, Sci. Sinica 24 (1981) 35, Damgaard-Huffel, Phys Rept 152 (1987) 227

You now want an extra degree of freedom which you will think of as a stochastic time in which an evolution takes place according to the Langevin equation

$$\phi(x) \mapsto \phi_\eta(x;t) \qquad \frac{d\phi_\eta(x;t)}{dt} = -\frac{\partial S[\phi]}{\partial \phi_\eta(x;t)} + \eta(x;t)$$

The drift term is given by the equations of motion...

... but beware! This is a stochastic differential equation due to the presence of the gaussian noise

$$\eta(x;t): \qquad \langle \eta(x,t) \;\, \eta(x',t') \rangle_\eta = 2\,\delta(x-x') \;\, \delta(t-t')$$

Noise expectation values are now naturally defined

$$\langle \ldots \rangle_\eta = \frac{\int D\eta(z,\tau) \;\, \ldots \; e^{-\frac{1}{4}\int dz d\tau \eta^2(z,\tau)}}{\int D\eta(z,\tau)\, e^{-\frac{1}{4}\int dz d\tau \eta^2(z,\tau)}}$$

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The key assertion of Stochastic Quantization can be now simply stated

$$\langle O[\phi_\eta(x_1;t)\ldots\phi_\eta(x_n;t)] \rangle_\eta \rightarrow_{t\rightarrow\infty} \langle O[\phi(x_1)\ldots\phi(x_n)] \rangle$$

A conceptually simple proof comes from the Fokker Planck equation formalism

$$\langle O[\phi_\eta(t)]\rangle_\eta = \frac{\int D\eta\, O[\phi_\eta(t)]\, e^{-\frac{1}{4}\int dz d\tau \eta^2(z,\tau)}}{\int D\eta\, e^{-\frac{1}{4}\int dz d\tau \eta^2(z,\tau)}} = \int D\phi\; O[\phi]\, P[\phi,t]$$

$$\dot{P}[\phi,t] = \int dx\; \frac{\delta}{\delta\phi(x)} \left(\frac{\delta S[\phi]}{\delta\phi(x)} + \frac{\delta}{\delta\phi(x)} \right) P[\phi,t]$$

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Floratos-Iliopoulos, Nucl.Phys. B 214 (1983) 392

for the solution of which we can introduce a perturbative expansion which generates a hierarchy of equations

$$P[\phi,t] = \sum_{k=0} g^k P_k[\phi,t]$$

Leading order is easy to solve and admits an infinite time (equilibrium) limit such that

$$P_0[\phi,t] {\rightarrow}_{t\rightarrow\infty} P_0^{eq}[\phi] = \frac{e^{-S_0[\phi]}}{Z_0}$$

In a convenient weak sense at every order one gets equilibrium $P_k[\phi,t] {\rightarrow}_{t\rightarrow\infty} P_k^{eq}[\phi]$

in terms of quantities which are interelated by a set of relations in which one recognizes the Schwinger-Dyson equations ... i.e. we are done!

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We want to go via another expansion, i.e. the expansion of the solution of Langevin equation in power of the coupling constant

$$\phi_\eta(x;t) = \phi_\eta^{(0)}(x;t) + \sum_{n>0} g^n \phi_\eta^{(n)}(x;t)$$

Parisi-Wu, Damgaard-Huffel

Langevin equation for the free scalar field (momentum space) $\dfrac{\partial}{\partial t}\phi_\eta^{(0)}(k,t) = -(k^2+m^2)\phi_\eta^{(0)}(k,t) + \eta(k,t)$

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i.e. $$G^{(0)}(k,t) = \theta(t)\,\exp\left(-(k^2+m^2)t\right)$$

$$\phi^{(0)}(k,t) = \phi^{(0)}(k,0)\exp\left(-(k^2+m^2)t\right) + \int_0^t d\tau \exp\left(-(k^2+m^2)(t-\tau)\right)\eta(k,\tau)$$

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Interacting case (cubic interaction in the following) is solved by superposition ...

$$\phi(k,t) = \int_0^t d\tau\exp -(k^2+m^2)(t-\tau)\left[\eta(k,\tau) - \frac{\lambda}{2!}\int\frac{dpdq}{(2\pi)^{2n}}\,\phi(p,\tau)\,\phi(q,\tau)\,\delta(k-p-q)\right]$$

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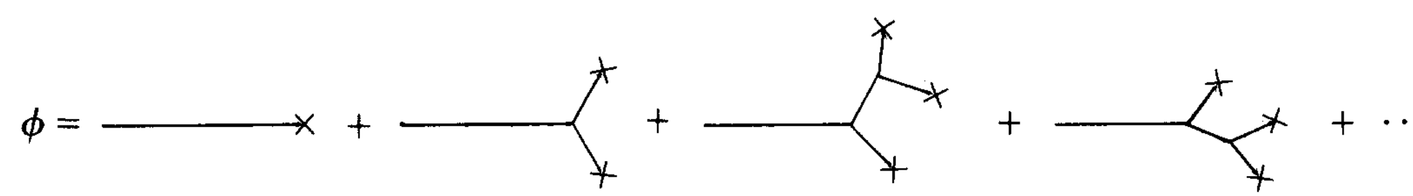
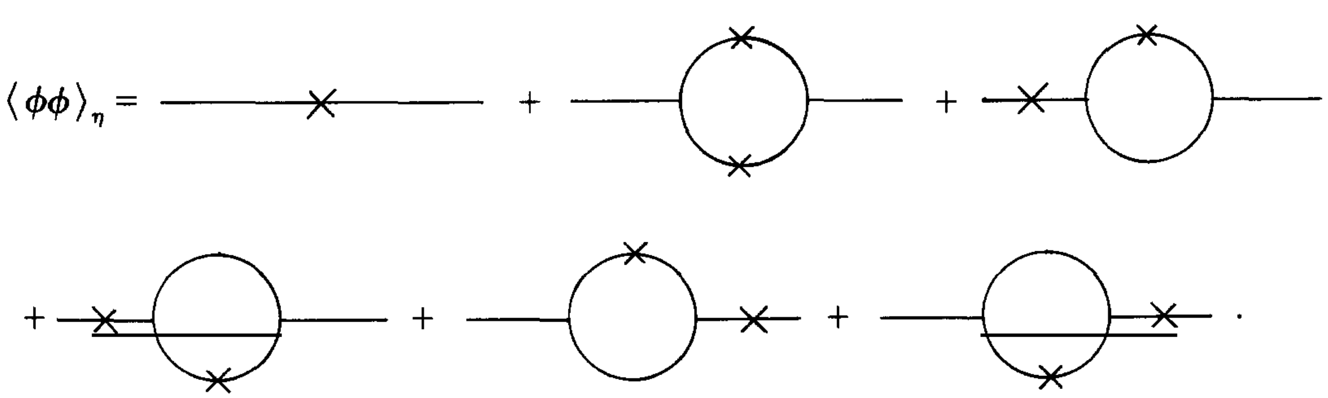
... which leaves the solution in a form which is ready for iteration. It is actually also ready for a graphical intepretation and for the formulation of a

diagrammatic Stochastic Perturbation Theory

$$\phi = \int G\eta - \frac{\lambda}{3!}\int\int\int\int G(G\eta)(G\eta) + \ldots$$





The stochastic diagrams one obtains when averaging over the noise (contractions!) reconstruct, in a convenient infinite time limit, the contributions of the (topologically) correspondent Feynman diagrams ...

but we do not want to go this way ...

NSPT in a nutshell Di Renzo, Marchesini, Onofri Nucl. Phys. B 426 (1994) 675

- Take $$\phi_\eta(x;t) = \phi_\eta^{(0)}(x;t) + \sum_{n>0} g^n \phi_\eta^{(n)}(x;t)$$

- Plug it into $$\frac{d\phi_\eta(x;t)}{dt} = -\frac{\partial S[\phi]}{\partial \phi_\eta(x;t)} + \eta(x;t)$$

- … which now becomes a HIERARCHY* of equations …

- … which you make the computer integrate for you!

* At any given order truncation is exact!

NSPT in plain English (coming back to the QM problem…)

QM via (lattice regularized) PATH INTEGRALS

(a primer by M. Creutz)

ANNALS OF PHYSICS 132, 427–462 (1981)

A Statistical Approach to Quantum Mechanics*

M. CREUTZ AND B. FREEDMAN

Physics Department, Brookhaven National Laboratory, Upton, New York 11973

Received November 13, 1980

A Monte Carlo method is used to evaluate the Euclidean version of Feynman's sum over particle histories. Following Feynman's treatment, individual paths are defined on a discrete (imaginary) time lattice with periodic boundary conditions. On each lattice site, a continuous position variable x_i specifies the spacial location of the particle. Using a modified Metropolis algorithm, the low-lying energy eigenvalues, $|\psi_0(x)|^2$, the propagator, and the effective potential for the anharmonic oscillator are computed, in good agreement with theory. For a deep double-well potential, instantons were found in our computer simulations appearing as multi-kink configurations on the lattice.

$$Z_{fi} = \langle x_f \,|\, e^{-HT/\hbar} \,|\, x_i \rangle \sim \int [dx]\, e^{-S[x]/\hbar},$$

$$S = \int_0^T d\tau \left[\tfrac{1}{2} m_0 \left[\frac{dx}{d\tau}\right]^2 + V(x)\right],$$

$$x(0) = x_i\,, \qquad x(T) = x_f$$

Put it on a LATTICE!

$$S = \sum_{j=1}^{N} a \left[\frac{1}{2} m_0 \frac{[x_{j+1} - x_j]^2}{a} + V(x_j)\right],$$

$$\langle \hat{A} \rangle = \mathrm{Tr}(e^{-HT/\hbar} \hat{A}) / \mathrm{Tr}(e^{-HT/\hbar}) = \langle 0 \,|\, \hat{A} \,|\, 0 \rangle, \qquad \text{as} \quad T \to \infty$$

or

$$\frac{\int_{-\infty}^{+\infty} \prod_{i=1}^{N} dx_i\, A(x_1, x_2, \ldots, x_n)\, e^{-1/\hbar S[x]}}{\int_{-\infty}^{+\infty} \prod_i dx_i\, e^{-1/\hbar S[x]}}$$

… to be sampled by Monte Carlo

$$E_0 = \lim_{T \to \infty} \left(\int [dx]\, e^{-1/\hbar S[x]} [\tfrac{1}{2} x V'(x) + V(x)] \Big/ \int [dx]\, e^{-1/\hbar S[x]} \right)$$

Suppose you want to compute the QUARTIC OSCILLATOR in NSPT

$$S[x] = \frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 + \frac{1}{2}m\omega^2 x^2 + \lambda x^4$$

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This is your expansion … $x = x^{(0)} + \lambda x^{(1)} + \lambda^2 x^{(2)} + \ldots$

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This is your expansion … $x = x^{(0)} + \lambda x^{(1)} + \lambda^2 x^{(2)} + \ldots$

… and these are your equations

$$\begin{cases} \frac{d}{dt_L}x^{(0)} = m\frac{d^2x^{(0)}}{d\tau^2} - m\omega^2x^{(0)} + \eta \\ \frac{d}{dt_L}x^{(1)} = m\frac{d^2x^{(1)}}{d\tau^2} - m\omega^2x^{(1)} + 4\,{x^{(0)}}^3 \\ \frac{d}{dt_L}x^{(2)} = m\frac{d^2x^{(2)}}{d\tau^2} - m\omega^2x^{(2)} + 4*3\,x^{(1)}{x^{(0)}}^2 \\ \ldots \end{cases}$$

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Needless to say

- You need to expand your observables as well
- You need a numerical integration scheme (Euler, Runge Kutta, …)

IT WORKS!

Of course there are smarter ways to compute this…

BenderWu.nb

```
VZero = 3/4 EigV[0] + 3 Sqrt[2]/2 EigV[2] + Sqrt[6]/2
EigV[4];

EigV[n_Integer] := 0 /; n<0;

Bracket[a__ EigV[n_Integer],c__] := a
Bracket[EigV[n],c];
Bracket[c__,a__ EigV[n_Integer]] := a
Bracket[c,EigV[n]];

Bracket[Plus[a_. EigV[n_Integer],b__],c__] :=
a Bracket[EigV[n],c] + Bracket[Plus[b],c];

Bracket[c__,Plus[a_. EigV[n_Integer],b__]] :=
a Bracket[c,EigV[n]] + Bracket[c,Plus[b]];

Bracket[EigV[n_Integer],EigV[m_Integer]] :=

If[n==m,1,0];

OpeR[x__] := OpeR[ExpandAll[x]];

OpeR[a__ EigV[n_Integer]] := a OpeR[EigV[n]];
OpeR[Plus[a_. EigV[n_Integer],b__]] := a OpeR[EigV[
OpeR[Plus[b]];
OpeR[EigV[n_Integer]] := 1/n EigV[n] /; n>0;
OpeR[EigV[0]] := 0;

OpeV[x__] := OpeV[ExpandAll[x]];

OpeV[a__ EigV[n_Integer]] := a OpeV[EigV[n]];
OpeV[Plus[a_. EigV[n_Integer],b__]] := a OpeV[EigV[
OpeV[Plus[b]];
OpeV[EigV[n_Integer]] :=
1/4(Sqrt[n(n-1)(n-2)(n-3)] EigV[n-4] +
(4n-2)*Sqrt[n(n-1)] EigV[n-2] +
3(2 n^2+2n+1) EigV[n] + (4n+6)*Sqrt[(n+1)(n+2)]
EigV[n+2] +
Sqrt[(n+1)(n+2)(n+3)(n+4)] EigV[n+4]);

deltaE[n_Integer] := deltaE[n] = Bracket[VZero,delt
1]];

deltaV[0] = EigV[0];
deltaV[1] = - OpeR[VZero];

deltaV[n_Integer] := deltaV[n] =
OpeR[Sum[deltaE[j] deltaV[n-j],{j,1,n-1}] -
OpeV[deltaV[n-1]]];
```

A few different frameworks for NSPT (i.e. a few handles to possibly improve it)

(there are projects going on this!)

There are various formulations of NSPT one can think of …

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(1) Is Langevin the only stochastic equation one can play with in NSPT?
NO! e.g. Stochastic Molecular Dynamics (SMD Horowitz 1985 …)

$$\frac{d\phi(x;t)}{dt} = \pi(x;t)$$
$$\frac{d\pi(x;t)}{dt} = -\frac{\partial S[\phi]}{\partial \phi(x;t)} - 2\,\mu_0 \pi(x;t) + \eta(x;t)$$

$$\eta(x;t): \quad \langle \eta(x,t)\ \eta(x',t') \rangle_\eta = 4\,\mu_0\,\delta(x-x')\,\delta(t-t')$$

which is Langevin for $\mu_0 \to \infty$

Notice that one can tune the lattice parameter $\gamma = 2\,\mu_0 a$ to minimize errors!
(which depend on both autocorrelation times and standard deviations (*)!)
(*) subtle issues in the continuum limit!

Dalla Brida Kennedy Garofalo 2015 Dalla Brida Luescher 2016 (Gradient Flow!)

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Dalla Brida Kennedy Garofalo 2015 Dalla Brida Luescher 2016 (Gradient Flow!)

(2) Numerical integrators (numerical integration schemes) DO MATTER!
… and of course various combinations are possible … e.g.

(2a) Langevin with 2nd order integrator

(2b) Stochastic Molecular Dynamics with 4th order OMF integrator

Bali Bauer Torrero 2008 Dalla Brida Kennedy Garofalo 2015 Dalla Brida Luescher 2016

Something in which NSPT can quite easily perform well

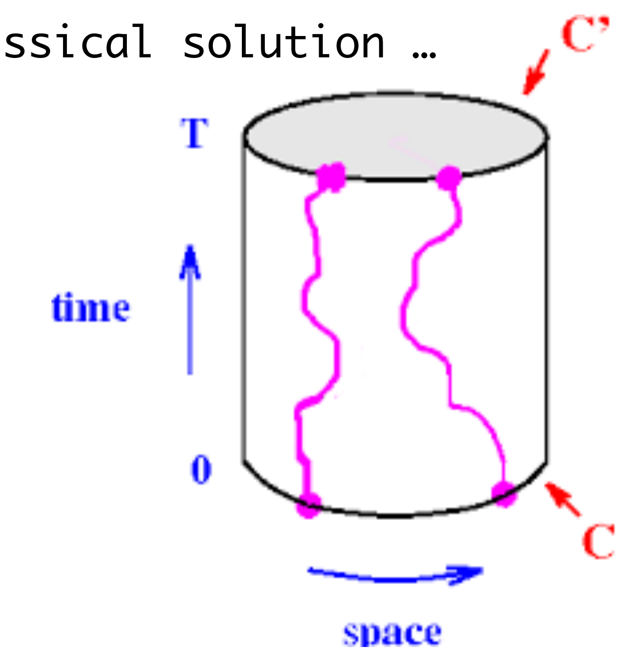
PERTURBATION THEORY IN THE BACKGROUND OF AN INSTANTON!

A canonical example of expansion around a non-trivial vacuum: the Schrodinger Functional (SF) in NSPT

Brambilla, Dalla Brida, Di Renzo, Hesse, Sint

The SF is a perfect framework for NSPT! Fluctuations in the background of classical solution …

$$U_k(x)|_{x_0=0} = e^{aC_k}\,, \quad U_k(x)|_{x_0=T} = e^{aC'_k}$$



$$U_\mu(x) = e^{a\,g_0\,q_\mu(x)} V_\mu(x)$$

Alpha Collaboration

$$V_\mu(x) = e^{a\,B_\mu(x)}\,, \quad B_0 = 0\,, \quad B_k(x) = \frac{1}{T}\left[x_0 C'_k + (T - x_0) C_k\right]$$

Define and compute the SF coupling

$$\Gamma = -\ln\left[\int D[U] e^{-S[U]}\right] \qquad \frac{k}{\overline{g}^2} = \left.\frac{\partial\Gamma}{\partial\eta}\right|_{\eta=\nu=0} \qquad \overline{g}^2 = g_0^2\,(1 + m_1\,g_0^2 + m_2\,g_0^4 + \dots)$$

We have a working implementation of the SF in NSPT!
At least, useful for taking the continuum limit …

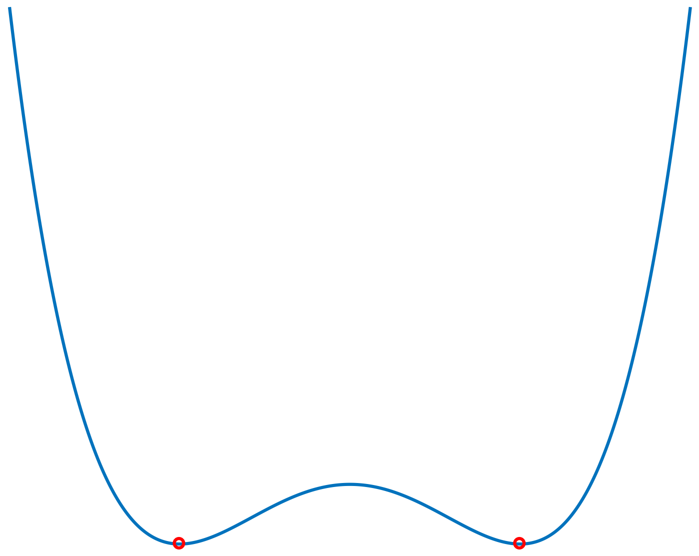
ALPHA Collaboration / Nuclear Physics B 713 (2005) 378–406

2.4. Discretization effects

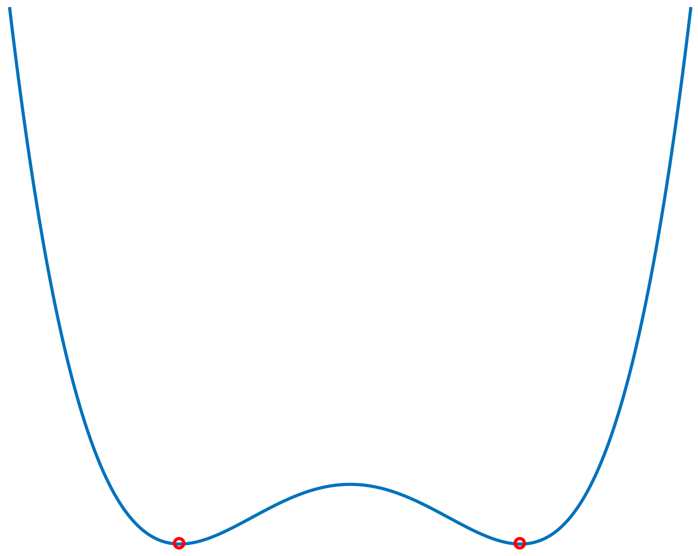
The influence of the underlying space–time lattice on the evolution of the coupling can be estimated perturbatively [29], by generalizing Symanzik's discussion [36–38] to the present case. Close to the continuum limit we expect that the relative deviation

$$\delta(u, a/L) = \frac{\Sigma(u, a/L) - \sigma(u)}{\sigma(u)} = \delta_1(a/L)u + \delta_2(a/L)u^2 + \cdots \qquad (2.30)$$

Let’s consider the DOUBLE WELL potential



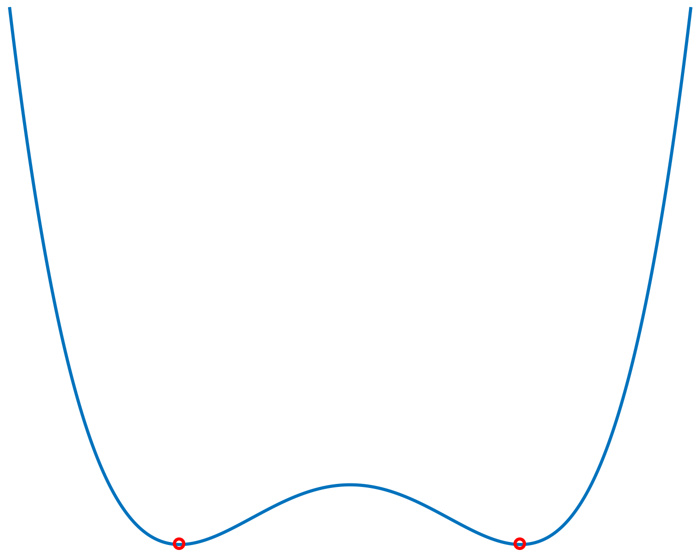
Let's consider the DOUBLE WELL potential



Around which vacuum are we going to expand?

Notice that till now we have always assumed the trivial one…

Let’s consider the DOUBLE WELL potential



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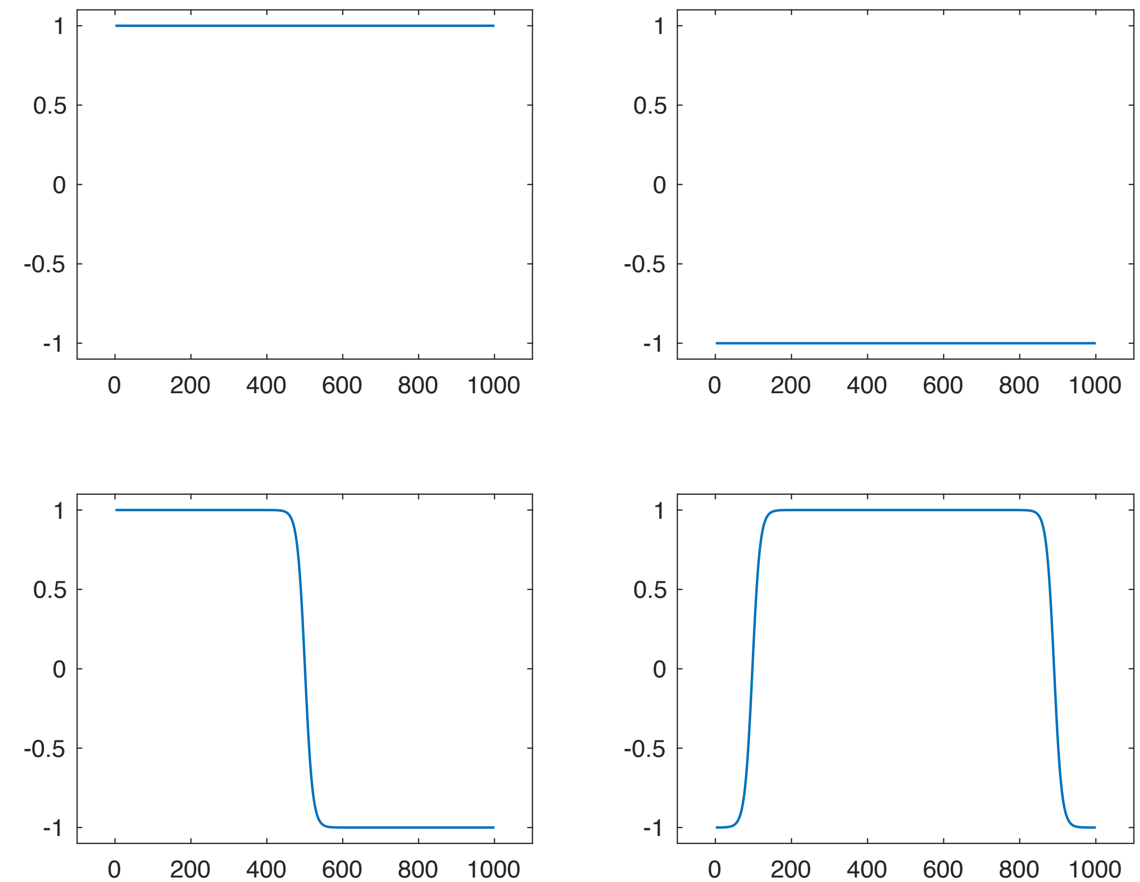
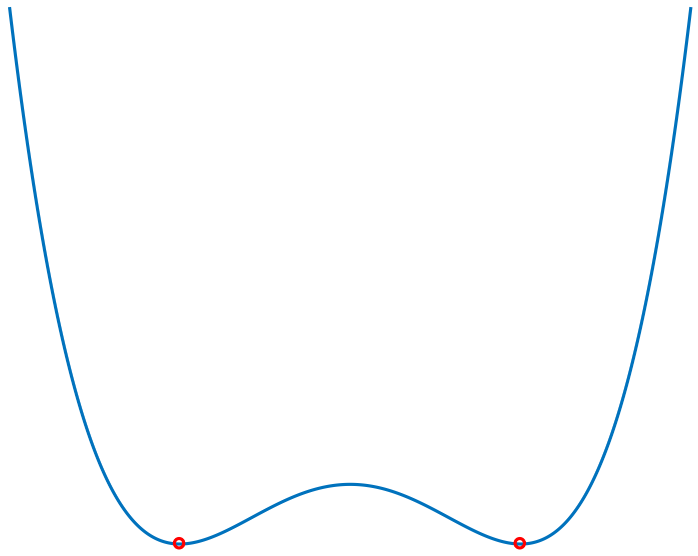


… better …

around which classical solution?

Let's consider the DOUBLE WELL potential



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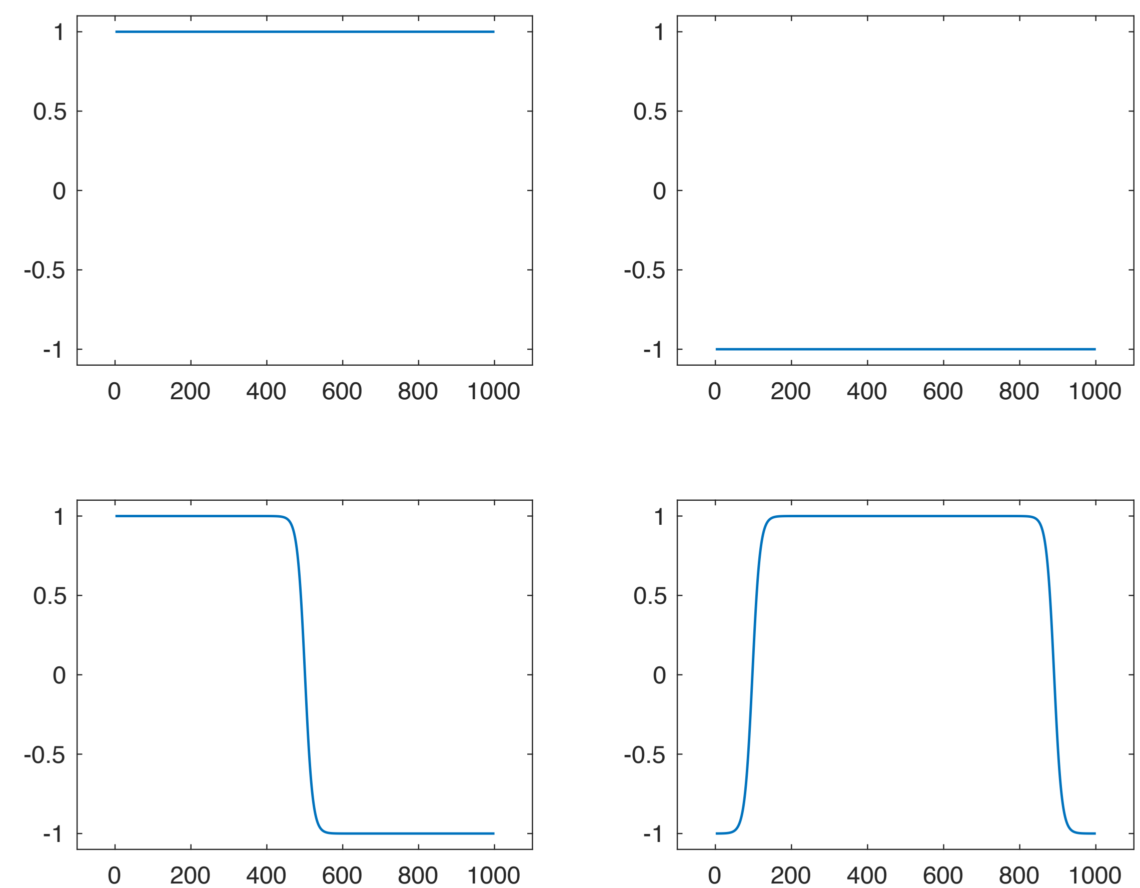


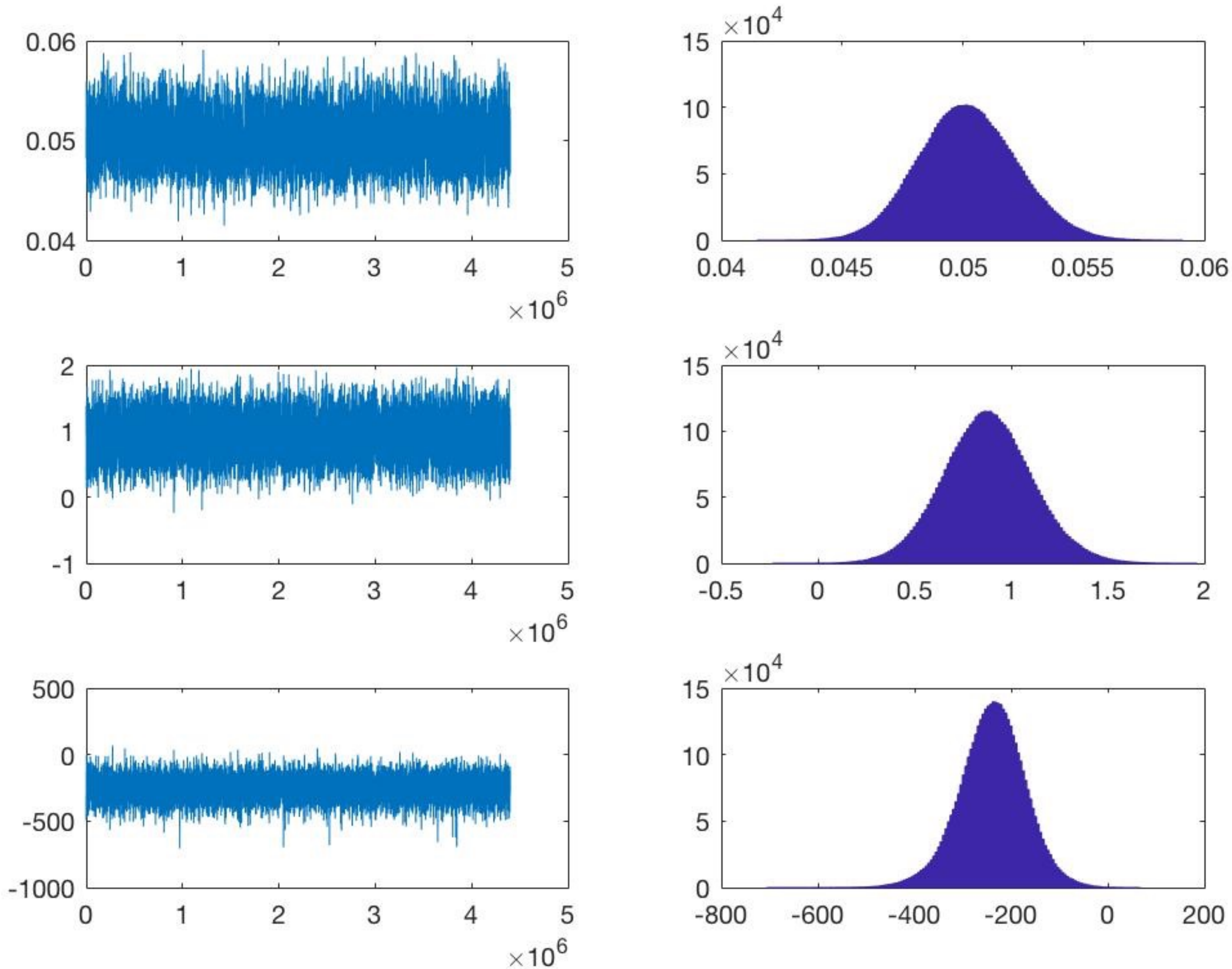
… better …

around which classical solution?

RECIPE

1. Select a classical solution x_{cl}
2. Re-express your field as $x = x_{cl} + x_{fluct}$
3. Plug this in and then write the NSPT expansion for x_{fluct}
4. You will get a PERTURBATIVE COMPUTATION AROUND (say) AN INSTANTON!

Apparently it works! These are signals for NSPT around an instanton for DW
Everything still very preliminary… just work of these days!



Conclusions

- NSPT has been around for roughly 20 years, but it is never too late to have a closer look at it!

- I think there can be many applications relevant for Resurgence!

- Remember: it is a numerical method (with issues of errors and statistics), but in many cases it is quite difficult to have something better than this at high orders!

… extra stuff …

Something maybe more field-theoretic (numerics stumbles on fundamental QFT…)

Renormalons

An old goal: a lattice determination of the gluon condensate …

… where an OPE is in place … $W = \langle \alpha_s\, F^2 \rangle / Q^4 \;=\; W_0 + (\Lambda^4/Q^4)\, W_4 + \cdots$

… now the plaquette is our observable $W(N) \;=\; 1 - \frac{1}{3}\, \langle \mathrm{Tr} U_p \rangle$

… unavoidably computed on a lattice of finite extent Na

Perturbative (PT) contribution (associated to the identity) should be subtracted from Non-Perturbative (NPT) Monte Carlo (MC) data measured at various values of the lattice coupling β , looking for the signature dictated by asymptotic scaling, i.e. $\Lambda a \sim e^{-\beta/12 b_0}$

$$W_{\rm MC} - W_{\rm pert} = (\Lambda^4/Q^4)\, W_4 + \cdots$$

PROBLEM: expect RENORMALONS!
From dimensional and RG arguments

$$W^{\rm ren} = C \int_{r\Lambda^2}^{Q^2} \frac{k^2\, dk^2}{Q^4}\, \alpha_s(k^2)$$

by changing variable $z \equiv z_0 \left(1 - \alpha_s(Q^2)/\alpha_s(k^2)\right)$ $\qquad z_0 \equiv \frac{1}{3 b_0}$

$$W^{\rm ren} = \mathcal{N} \int_0^{z_{0\_}} dz\; e^{-\beta z}\, (z_0 - z)^{-1-\gamma}$$

The experts will recognize a Borel integral …

$$4\pi\alpha_s(Q^2) \equiv 6/\beta \qquad \gamma \equiv 2\frac{b_1}{b_0^2} \qquad 0 < z < z_{0\_} \equiv z_0(1 - \alpha_s(Q^2)/\alpha_s s(r\Lambda^2))$$

$$W^{\rm ren} = \sum_{\ell=1} \beta^{-\ell}\, \{c_\ell^{ren} + \mathcal{O}(e^{-z_0 \beta})\} \qquad c_\ell^{ren} = \mathcal{N}'\, \Gamma(\ell+\gamma)\, z_0^{-\ell}$$

CAN WE INSPECT RENORMALONS IN A NSPT COMPUTATION OF THE PLAQUETTE? Di Renzo Marchesini Onofri 1995

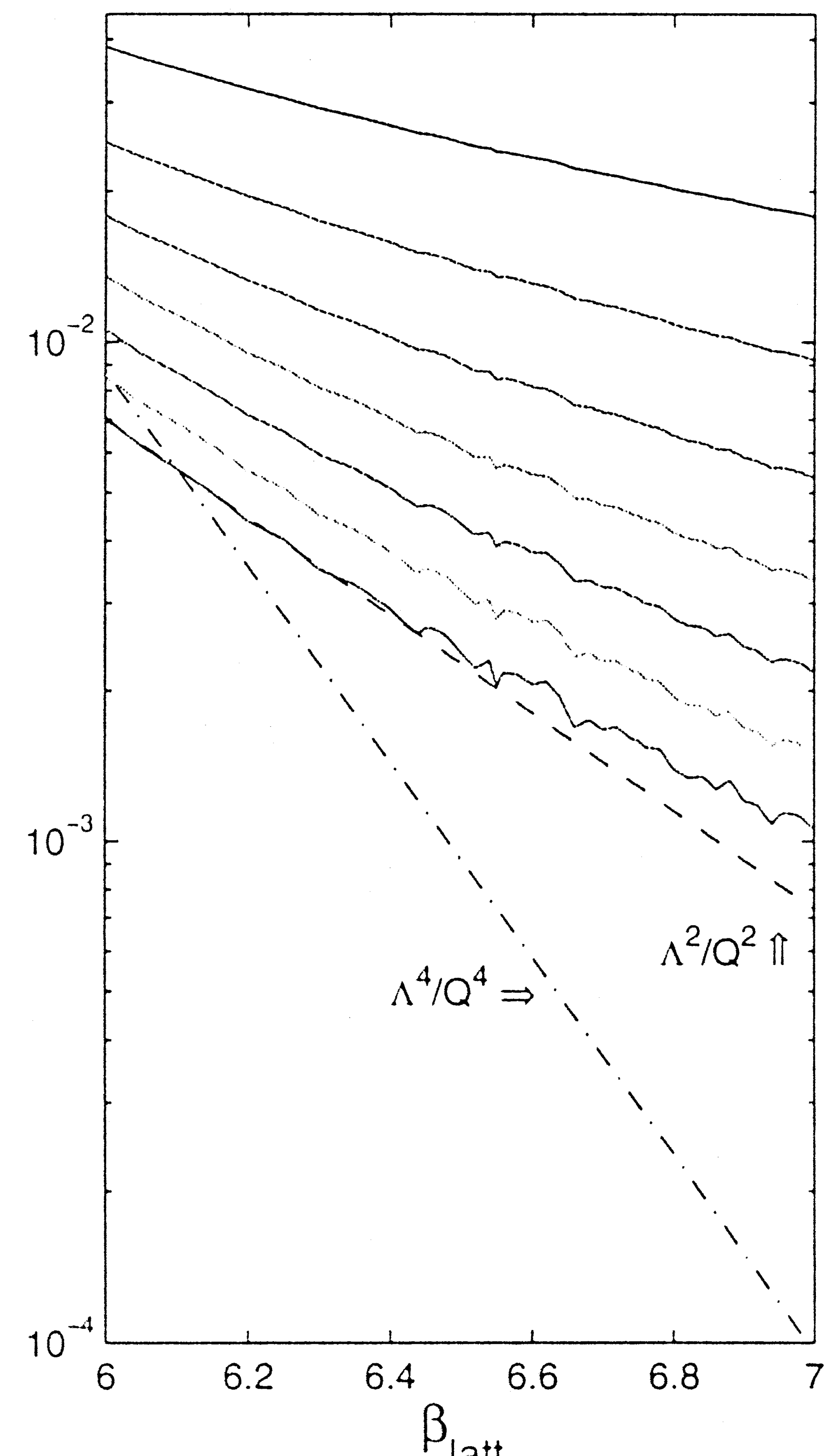
PROBLEMS

1. Computing power …
2. The IR renormalon deserves its name and relevant momenta go like $k^* \sim s^{-1} e^{-(\ell-1)/2}$

… rather study $$W^{\rm ren}(N) = C \int_{Q_0^2(N)}^{Q^2} \frac{k^2\, dk^2}{Q^4}\, \alpha_s(sk^2) \quad \to \quad \sum_{\ell=1} \beta^{-\ell}\, c_\ell^{ren}(N; s, C)$$

… where the finite lattice has been explicitly taken into account, while the change of scale can be reabsorbed in a change of scheme (i.e., look for a scheme in which renormalon is better described…)

… but all in all the final result for the subtraction was signaling something odd going on … WRONG SCALING!
Burgio Di Renzo Marchesini Onofri 1998



We now know that NSPT CAN ACTUALLY DIRECTLY INSPECT RENORMALONS, but one has to go to HIGHER ORDERS … (at the time the first 8 orders had been computed)

Solution of the puzzle and direct inspection of renormalons Bali Bauer Pineda 2014

In 2012, Horsley et al computed the first 20 orders.

In 2013 Bali and Pineda detected the renormalon in the HQET/pole mass framework: dimensions do matter! The order at which renormalons show up increases with the dimension of the operator!

Improvements (Bali Pineda) for the plaquette case (2014):

1. Twisted BCs (which kill zero modes; I have cheated a little bit about those till now…)
2. 2nd order integrator for Langevin equation(s)
3. computer power (well … it was 20 years later …)
4. careful treatment of finite size effects by perturbative OPE (separation of scales!)

$$\langle P\rangle_{\rm pert}(N) = P_{\rm pert}(\alpha)\langle \mathbb{1}\rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4 \langle O_{\rm G}\rangle_{\rm soft} + \mathcal{O}\left(\frac{1}{N^6}\right) \qquad \frac{1}{a} \gg \frac{1}{Na}$$

$$P_{\rm pert}(\alpha) = \sum_{n\geq 0} p_n \alpha^{n+1} \qquad \frac{\pi^2}{36}\, a^4 \langle O_{\rm G}\rangle_{\rm soft} = -\frac{1}{N^4}\sum_{n\geq 0} f_n \alpha^{n+1}((Na)^{-1})$$

with both p_n and f_n asymptotically dominated by the IR renormalon!

Normalization for Wilson action is fixed by $C_{\rm G}(\alpha) = 1 + \sum_{k\geq 0} c_k \alpha^{k+1} = -\frac{\beta_0 \alpha^2}{2\pi\beta(\alpha)}$

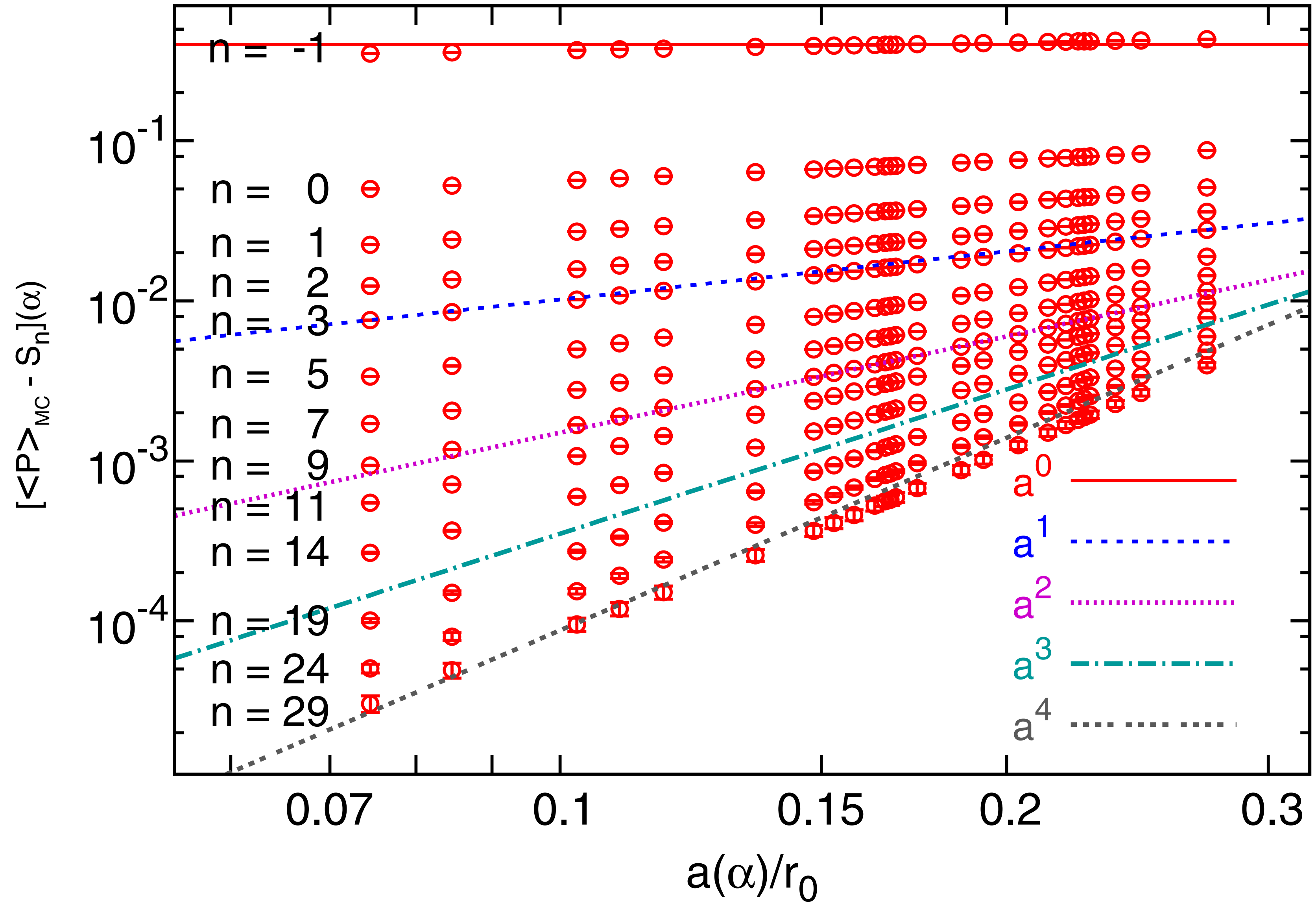
… and one can finally fit the computed $\langle P\rangle_{\rm pert}(N) = \sum_{n\geq 0}\left[p_n - \frac{f_n(N)}{N^4}\right]\alpha^{n+1}$

IT WORKS!

	$c_n^{(3,0)}$	$c_n^{(3,1/6)}$	$c_n^{(8,0)} C_F/C_A$	$c_n^{(8,1/6)} C_F/C_A$
c_0	2.117274357	0.72181(99)	2.117274357	0.72181(99)
c_1	11.136(11)	6.385(10)	11.140(12)	6.387(10)
$c_2/10$	8.610(13)	8.124(12)	8.587(14)	8.129(12)
$c_3/10^2$	7.945(16)	7.670(13)	7.917(20)	7.682(15)
$c_4/10^3$	8.215(34)	8.017(33)	8.197(42)	8.017(36)
$c_5/10^4$	9.322(59)	9.160(59)	9.295(76)	9.139(64)
$c_6/10^6$	1.153(11)	1.138(11)	1.144(13)	1.134(12)
$c_7/10^7$	1.558(21)	1.541(22)	1.533(25)	1.535(22)
$c_8/10^8$	2.304(43)	2.284(45)	2.254(51)	2.275(45)
$c_9/10^9$	3.747(95)	3.717(97)	3.64(11)	3.703(98)
$c_{10}/10^{10}$	6.70(22)	6.65(22)	6.49(25)	6.63(22)
$c_{11}/10^{12}$	1.316(52)	1.306(53)	1.269(59)	1.303(53)
$c_{12}/10^{13}$	2.81(13)	2.79(13)	2.71(14)	2.78(13)
$c_{13}/10^{14}$	6.51(35)	6.46(35)	6.29(37)	6.45(35)
$c_{14}/10^{16}$	1.628(96)	1.613(97)	1.57(10)	1.614(97)
$c_{15}/10^{17}$	4.36(28)	4.32(28)	4.22(29)	4.33(28)
$c_{16}/10^{19}$	1.247(86)	1.235(86)	1.206(89)	1.236(86)
$c_{17}/10^{20}$	3.78(28)	3.75(28)	3.66(28)	3.75(28)
$c_{18}/10^{22}$	1.215(93)	1.204(94)	1.176(95)	1.205(94)
$c_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

	$f_n^{(3,0)}$	$f_n^{(3,1/6)}$	$f_n^{(8,0)} C_F/C_A$	$f_n^{(8,1/6)} C_F/C_A$
f_0	0.7696256328	0.7810(59)	0.7696256328	0.7810(69)
f_1	6.075(78)	6.046(58)	6.124(87)	6.063(68)
$f_2/10$	5.628(91)	5.644(62)	5.60(11)	5.691(78)
$f_3/10^2$	5.87(11)	5.858(76)	6.00(18)	5.946(91)
$f_4/10^3$	6.33(22)	6.29(17)	6.57(40)	6.26(23)
$f_5/10^4$	7.73(35)	7.71(26)	7.67(66)	7.78(42)
$f_6/10^5$	9.86(53)	9.80(42)	9.68(99)	9.79(69)
$f_7/10^7$	1.388(81)	1.378(71)	1.35(15)	1.38(11)
$f_8/10^8$	2.12(12)	2.11(12)	2.06(22)	2.10(17)
$f_9/10^9$	3.54(20)	3.52(20)	3.40(37)	3.51(27)
$f_{10}/10^{10}$	6.49(33)	6.44(34)	6.23(67)	6.44(43)
$f_{11}/10^{12}$	1.296(64)	1.286(66)	1.24(13)	1.286(74)
$f_{12}/10^{13}$	2.68(19)	2.64(18)	2.65(33)	2.65(21)
$f_{13}/10^{14}$	6.70(54)	6.68(52)	6.36(90)	6.66(57)
$f_{14}/10^{16}$	1.58(14)	1.56(14)	1.55(22)	1.57(15)
$f_{15}/10^{17}$	4.41(34)	4.37(33)	4.24(47)	4.37(35)
$f_{16}/10^{19}$	1.241(92)	1.230(91)	1.20(11)	1.231(94)
$f_{17}/10^{20}$	3.79(28)	3.75(28)	3.67(30)	3.76(28)
$f_{18}/10^{22}$	1.215(94)	1.204(94)	1.176(97)	1.205(94)
$f_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

… and they could finally
determine the gluon condensate



PRL **113,** 092001 (2014) PHYSICAL REVIEW LETTERS week ending 29 AUGUST 2014

Model Independent Determination of the Gluon Condensate in Four Dimensional SU(3) Gauge Theory

Gunnar S. Bali,[1,2] Clemens Bauer,[1] and Antonio Pineda[3]
[1]*Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany*
[2]*Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India*
[3]*Grup de Física Teòrica, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Barcelona, Spain*
(Received 25 March 2014; revised manuscript received 7 May 2014; published 25 August 2014)

We determine the nonperturbative gluon condensate of four-dimensional SU(3) gauge theory in a model-independent way. This is achieved by carefully subtracting high-order perturbation theory results from nonperturbative lattice QCD determinations of the average plaquette. No indications of dimension-two condensates are found. The value of the gluon condensate turns out to be of a similar size as the intrinsic ambiguity inherent to its definition. We also determine the binding energy of a *B* meson in the heavy quark mass limit.

DOI: 10.1103/PhysRevLett.113.092001 PACS numbers: 12.38.Gc, 11.55.Hx, 12.38.Bx, 12.38.Cy

… extra extra stuff …

Stochastic Quantization for LGT Batrouni et al (Cornell group) PRD 32 (1985)

We now start with the Wilson action $$S_G = -\frac{\beta}{2N_c}\sum_P \mathrm{Tr}\left(U_P + U_P^\dagger\right)$$

We now deal with a theory formulated in terms of group variables and Langevin equation reads

$$U_{\mu x} = e^{A_\mu(x)}$$

$$\frac{\partial}{\partial t}U_{x\mu}(t;\eta) = \left(-i\nabla_{x\mu}S_G[U] - i\eta_{x\mu}(t)\right)U_{x\mu}(t;\eta)$$

where the Lie derivative is in place

$$\nabla_{x\mu} = T^a\nabla^a_{x\mu} = T^a\nabla^a_{U_{x\mu}} \qquad \nabla^a_V f(V) = \lim_{\alpha\to 0}\frac{1}{\alpha}\left(f\left(e^{i\alpha T^a}V\right) - f(V)\right)$$

This is again a stochastic differential equation with (gaussian) noise averages satisfying

$$\lim_{t\to\infty}\langle O[U(t;\eta)]\rangle_\eta = \frac{1}{Z}\int DU\, e^{-S_G[U]}\,O[U]$$

In order to proceed we now need a (numerical) integration scheme to simulate, e.g. Euler

$$U_{x\mu}(n+1;\eta) = e^{-F_{x\mu}[U,\eta]}\,U_{x\mu}(n;\eta) \qquad F_{x\mu}[U,\eta] = \epsilon\nabla_{x\mu}S_G[U] + \sqrt{\epsilon}\,\eta_{x\mu}$$

$$F_{x\mu}[U,\eta] = \frac{\epsilon\beta}{4N_c}\sum_{U_P\supset U_{x\mu}}\left[\left(U_P - U_P^\dagger\right) - \frac{1}{N_c}\mathrm{Tr}\left(U_P - U_P^\dagger\right)\right] + \sqrt{\epsilon}\,\eta_{x\mu}$$

$$\langle \eta_{i,k}(z)\,\eta_{l,m}(w)\rangle_\eta = \left[\delta_{il}\,\delta_{km} - \frac{1}{N_c}\,\delta_{ik}\,\delta_{lm}\right]\delta_{zw}$$

NSPT (directly in the LGT case) Di Renzo, Marchesini, Onofri 94

Now we look for a solution in the form of a perturbative expansion

$$U_{x\mu}(t;\eta) \to 1 + \sum_{k=1} \beta^{-k/2} U_{x\mu}^{(k)}(t;\eta)$$

then we plug it into the (numerical scheme!) Langevin equation and get a hierarchy of equations!

$$U^{(1)'} = U^{(1)} - F^{(1)}$$

$$U^{(2)'} = U^{(2)} - F^{(2)} + \frac{1}{2} F^{(1)\,2} - F^{(1)} U^{(1)}$$

$$U^{(3)'} = U^{(3)} - F^{(3)} + \frac{1}{2}(F^{(2)}F^{(1)} + F^{(1)}F^{(2)}) - \frac{1}{3!} F^{(1)\,3} - (F^{(2)} - \frac{1}{2} F^{(1)\,2})\, U^{(1)} - F^{(1)} U^{(2)}$$

$$\ldots$$

In practice: we do not look closely at the (underlying) Stochastic Perturbation Theory because the computer is going to (numerically) take care of it and all that you are interested in are the observables, for which

$$\langle O[\sum_k g^k \phi_\eta^{(k)}(t)] \rangle_\eta = \sum_k g^k \langle O_k(t) \rangle_\eta \qquad \lim_{t\to\infty} \langle O_k(t) \rangle_\eta = \lim_{T\to\infty} 1/T \sum_{j=1}^{T} O_k(jn)$$

Beware! Lattice PT is (always!) a decompactification of lattice formulation, so that ultimately one should be able to make contact with the continuum Langevin equation, i.e.

$$\frac{\partial}{\partial t} A_\mu^a(\eta, x; t) \; = \; D_\nu^{ab} F_{\nu\mu}^b(\eta, x; t) + \eta_\mu^a(x; t)$$

Where has this gone?

We did not loose anything, since we can always think of all this in the algebra

$$A_{x\mu}(t;\eta) \to \sum_{k=1} \beta^{-k/2} A_{x\mu}^{(k)}(t;\eta)$$

$$A = \log(U) = \log\left(1 + \sum_{k>0} \beta^{-\frac{k}{2}} U^{(k)}\right)$$

$$= \frac{1}{\sqrt{\beta}} U^{(1)} + \frac{1}{\beta}\left(U^{(2)} - \frac{1}{2} U^{(1)\,2}\right) + \left(\frac{1}{\beta}\right)^{\frac{3}{2}} \left(U^{(3)} - \frac{1}{2}\left(U^{(1)}U^{(2)} + U^{(2)}U^{(1)}\right) + \frac{1}{3} U^{(1)\,3}\right) + \dots$$

$$= \frac{1}{\sqrt{\beta}} A^{(1)} + \frac{1}{\beta} A^{(2)} + \left(\frac{1}{\beta}\right)^{\frac{3}{2}} A^{(3)} + \dots \qquad A^{(k)\,\dagger} = -A^{(k)} \qquad \mathrm{Tr} A^{(k)} = 0 \qquad \forall k$$

and the (expanded) Langevin equation now reads

$$A^{(1)'} = A^{(1)} - F^{(1)}$$

$$A^{(2)'} = A^{(2)} - F^{(2)} - \frac{1}{2}\left[F^{(1)}, A^{(1)}\right]$$

$$A^{(3)'} = A^{(3)} - F^{(3)} - \frac{1}{2}\left[F^{(1)}, A^{(2)}\right] - \frac{1}{2}\left[F^{(2)}, A^{(1)}\right] + \frac{1}{12}\left[F^{(1)}, \left[F^{(1)}, A^{(1)}\right]\right] + \frac{1}{12}\left[A^{(1)}, \left[F^{(1)}, A^{(1)}\right]\right]$$

... which I wanted to specify because it is an effective way of preparing for the fact that this is not the end of the story! Problems are going to pop out which we have to take care of ...

Stochastic Gauge Fixing

Stochastic Gauge Fixing D. Zwanziger, Nucl.Phys. B 192 (1981) 259

Let's go back to the continuum

$$\frac{\partial}{\partial t} A^a_\mu(\eta,x;t) \;=\; D^{ab}_\nu F^b_{\nu\mu}(\eta,x;t) + \eta^a_\mu(x;t)$$

whose expanded version has a (momentum space) solution

$$A^{(n)a}_\mu(k;t) \;=\; T^{ab}_{\mu\nu}\int_0^t ds\, e^{-k^2(t-s)} f^{(n)b}_\nu(k,s) + L^{ab}_{\mu\nu}\int_0^t ds\, f^{(n)b}_\nu(k,s)$$

in which vertices pop in (as they should ...)

$$f^{(0)a}_\nu(k;t) = \eta_\nu(k;t)^a \qquad\qquad f^{(n)a}_\nu(k;t) = gI^{(3)(n-1)a}_\mu(k;t) + g^2 I^{(4)(n-2)a}_\mu(k;t)$$

$$gI^{(3)a}_\mu(k;t) \;=\; \frac{igf^{abc}}{2(2\pi)^n}\int dpdq\,\delta(k+p+q)\,A^b_\nu(-p;t)\,A^c_\sigma(-q;t)\,v^{(3)}_{\mu\nu\sigma}(k,p,q)$$

$$v^{(3)}_{\mu\nu\sigma}(k,p,q) \;=\; \delta_{\mu\nu}(k-p)_\sigma + \text{cyclic permutations}$$

Remember the scalar case ...

$$\phi(k,t) = \int_0^t d\tau \exp -(k^2+m^2)(t-\tau)\left[\eta(k,\tau) - \frac{\lambda}{3!}\int \frac{dpdqds}{(2\pi)^{2n}}\,\phi(p,\tau)\,\phi(q,\tau)\,\phi(s,\tau)\,\delta(k-p-q-s)\right]$$

BUT ALL THIS IS GOING TO BE ONLY FORMAL ... WE WILL NOT OBTAIN LONG TIME CONVERGENCE BECAUSE OF THE LOSS OF DAMPING IN THE LONGITUDINAL (NON-gauge-invariant) SECTOR

SOLUTION: add an extra piece $$\dot{A}^a_\mu(x;t) = -\frac{\delta S[A]}{\delta A^a_\mu(x;t)} - D^{ab}_\mu V^b[A,t] + \eta^a_\mu(x;t)$$

Any functional evolves like $$\frac{\partial F[A]}{\partial t} = \int dx \, \frac{\delta F[A]}{\delta A^a_\mu(x;t)} \frac{\partial A^a_\mu(x;t)}{\partial t}$$

but GAUGE INVARIANT ones are such that $$D^{ab}_\mu \frac{\delta F[A]}{\delta A^b_\mu(x)} = 0$$

and thus physics is unaffected! (integration by parts ...) ... while if we make a convenient choice for the extra term we have new damping factors in place!

$$-D^{ab}_\mu V^b = \frac{1}{\alpha} D^{ab}_\mu \partial_\nu A^b_\nu$$

$$A^{a(n)}_\mu(k;t) = T_{\mu\nu} \int_0^t ds \, e^{-k^2(t-s)} f^{a(n)}_\nu(k,s) + L_{\mu\nu} \int_0^t ds \, e^{-\frac{k^2}{\alpha}(t-s)} f^{a(n)}_\nu(k,s)$$

On the lattice we interleave a gauge fixing step to the Langevin evolution

$$U'_{x\mu} = e^{-F_{x\mu}[U,\eta]} \, U_{x\mu}(n)$$

$$U_{x\mu}(n+1) = e^{w_x[U']} \, U'_{x\mu} \, e^{-w_{x+\hat{\mu}}[U']}$$

which has by the way an obvious interpretation

$$U_{x\mu}(n+1) = e^{-F_{x\mu}[U^G, G\eta G^\dagger]} \, U^G_{x\mu}(n)$$

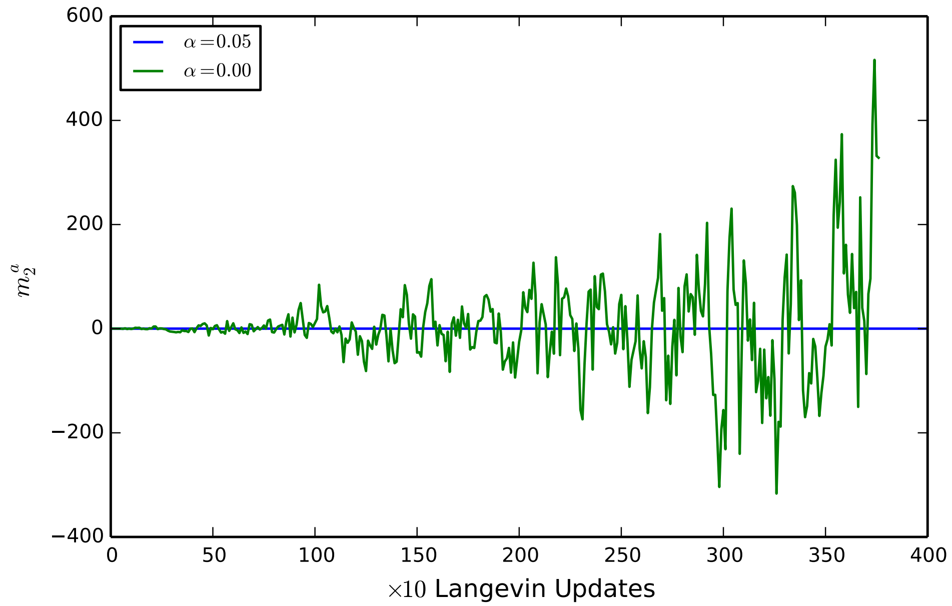


Figure 1. The effect of stochastic gauge fixing.

Fermionic loops in NSPT

FERMIONIC LOOPS in NSPT Di Renzo, Scorzato 2001

Let's add fermions (Wilson fermions, in this case) in the Langevin equation

$$S_F^{(W)} = \sum_{xy} \bar{\psi}_x \, M_{xy}[U] \, \psi_y$$
$$= \sum_x (m+4) \, \bar{\psi}_x \, \psi_x - \frac{1}{2} \sum_{x\mu} \left(\bar{\psi}_{x+\hat{\mu}} \, (1+\gamma_\mu) \, U^\dagger_{x\mu} \, \psi_x \; + \; \bar{\psi}_x \, (1-\gamma_\mu) \, U_{x\mu} \, \psi_{x+\hat{\mu}} \right)$$

From the point of view of the functional integral measure $e^{-S_G} \det M = e^{-S_{eff}} = e^{-(S_G - Tr \ln M)}$

and in turns $\nabla^a_{x\mu} S_G \mapsto \nabla^a_{x\mu} S_{eff} = \nabla^a_{x\mu} S_G - \nabla^a_{x\mu} \mathrm{Tr} \, \ln M = \nabla^a_{x\mu} S_G - \mathrm{Tr} \, ((\nabla^a_{x\mu} M) M^{-1})$

Batrouni et al (Cornell group) PRD 32 (1985)

In $U_{x\mu}(n+1;\eta) = e^{-F_{x\mu}[U,\eta]} \, U_{x\mu}(n;\eta)$ we now write

$$F = T^a (\epsilon \Phi^a + \sqrt{\epsilon} \eta^a) \qquad \Phi^a = \left[\nabla^a_{x\mu} S_G - \mathrm{Re} \left({\xi_k}^\dagger (\nabla^a_{x\mu} M)_{kl} (M^{-1})_{ln} \xi_n \right) \right]$$

where $\langle \xi_i \xi_j \rangle_\xi = \delta_{ij}$ or (this is what we always do)

$$\Phi^a = \left[\nabla^a_{x\mu} S_G - \mathrm{Re} \left({\xi_l}^\dagger (\nabla^a_{x\mu} M)_{ln} \psi_n \right) \right] \qquad M_{kl} \psi_l = \xi_k$$

From a numerical point of view this boils down to the (technically challenging) problem of inverting the Dirac operator efficiently. This is a heavy task, making unquenched simulations much more demanding in terms of computer time.

But we have not put our expansion in the coupling in place! Once we do it, we find much less problems than expected from the non-perturbative simulations point of view!

$$M = M^{(0)} + \sum_{k>0} \beta^{-k/2} M^{(k)} \qquad M^{-1} = {M^{(0)}}^{-1} + \sum_{k>0} \beta^{-k/2} {M^{-1}}^{(k)}$$

In NSPT we have to deal with only one inverse (known once and for all: the Feynman free propagator) plus a tower of recursive relations

$$\begin{aligned}
{M^{-1}}^{(1)} &= -{M^{(0)}}^{-1} M^{(1)} {M^{(0)}}^{-1} \\
{M^{-1}}^{(2)} &= -{M^{(0)}}^{-1} M^{(2)} {M^{(0)}}^{-1} - {M^{(0)}}^{-1} M^{(1)} {M^{-1}}^{(1)} \\
{M^{-1}}^{(3)} &= -{M^{(0)}}^{-1} M^{(3)} {M^{(0)}}^{-1} - {M^{(0)}}^{-1} M^{(2)} {M^{-1}}^{(1)} - {M^{(0)}}^{-1} M^{(1)} {M^{-1}}^{(2)}
\end{aligned}$$

i.e.

$${M^{-1}}^{(n)} = -{M^{(0)}}^{-1} \sum_{j=0}^{n-1} M^{(n-j)} {M^{(j)}}^{-1}$$

This has a direct counterpart in the solution of the linear system we have to face, which is also translated into a perturbative version (beware! the noise source is 0-th order)

$$\psi^{(j)} \equiv {M^{-1}}^{(j)} \xi$$

$$\begin{aligned}
\psi^{(0)} &= {M^{(0)}}^{-1} \xi \\
\psi^{(1)} &= -{M^{(0)}}^{-1} M^{(1)} \psi^{(0)} \\
\psi^{(2)} &= -{M^{(0)}}^{-1} \left[M^{(2)} \psi^{(0)} + M^{(1)} \psi^{(1)} \right] \\
\psi^{(3)} &= -{M^{(0)}}^{-1} \left[M^{(3)} \psi^{(0)} + M^{(2)} \psi^{(1)} + M^{(1)} \psi^{(2)} \right]
\end{aligned}$$

i.e.

$$\psi^{(n)} = -{M^{(0)}}^{-1} \sum_{j=0}^{n-1} M^{(n-j)} \psi^{(j)}$$

with ${M^{(0)}}^{-1}$ the (tree-level, field independent) Feynman propagator

which is particularly nice, since it can be solved by going back and forth from momentum to coordinate representation!

- A canonical application: renormalization constants

Renormalization constants used to be the realm of LPT …

… but these days this is NOT the case. A non-perturbative determination (where possible) is now the preferred choice (RI-MOM Rome group, SF ALPHA 90s). Still,

Renormalization is strictly speaking proved in PT
There are different systematics involved in PT and non-PT
… and at some point PT is supposed to converge (this is a UV problem …)

The RI-MOM schemes (Rome group 1994) are a good framework (in the massless limit). Being the scheme Regulator Independent, the coefficients of the logs are known! … and the finite parts are the easy part in NSPT …

Let's see how it works for quark bilinear (currents)

$$G_\Gamma(p) = \int dx\, \langle p|\, \overline{\psi}(x)\Gamma\psi(x)\, |p\rangle \qquad \Gamma_\Gamma(p) = S^{-1}(p)\, G_\Gamma(p)\, S^{-1}(p) \qquad O_\Gamma(p) = Tr\left(\hat{P}_{O_\Gamma}\Gamma_\Gamma(p)\right)$$

$$Z_{O_\Gamma}(\mu,\alpha) Z_q^{-1}(\mu,\alpha) O_\Gamma(p)|_{p^2=\mu^2} = 1 \qquad Z_q(\mu,\alpha) = -i\frac{1}{12}\frac{Tr(\not{p}S^{-1}(p))}{p^2}|_{p^2=\mu^2}$$

We know what to expect $\quad Z(\mu,\alpha_0) = 1 + \sum_{n>0} d_n(l)\,\alpha_0^n \qquad d_n(l) = \sum_{i=0}^{n} d_n^{(i)} l^i \qquad l \equiv \log(\mu a)^2$

A key ingredient is the quark 2-points function (beware! we will work with Wilson fermions…)

$$a\Gamma_2(\hat{p},\hat{m}_{cr},\beta^{-1}) = i\hat{\not{p}} + \hat{m}_W(\hat{p}) - \hat{\Sigma}(\hat{p},\hat{m}_{cr},\beta^{-1})$$

$$\hat{\Sigma}(\hat{p},\hat{m}_{cr},\beta^{-1}) = \hat{\Sigma}_c(\hat{p},\hat{m}_{cr},\beta^{-1}) + \hat{\Sigma}_\gamma(\hat{p},\hat{m}_{cr},\beta^{-1}) + \hat{\Sigma}_{\text{other}}(\hat{p},\hat{m}_{cr},\beta^{-1}) \qquad \left(\beta^{-1} \equiv \frac{2\pi\alpha_0}{3}\right)$$

$$\frac{1}{4}\sum_\mu \gamma_\mu \mathrm{Tr}_{\text{spin}}(\gamma_\mu\hat{\Sigma}) = \hat{\Sigma}_\gamma$$

What one really computes is

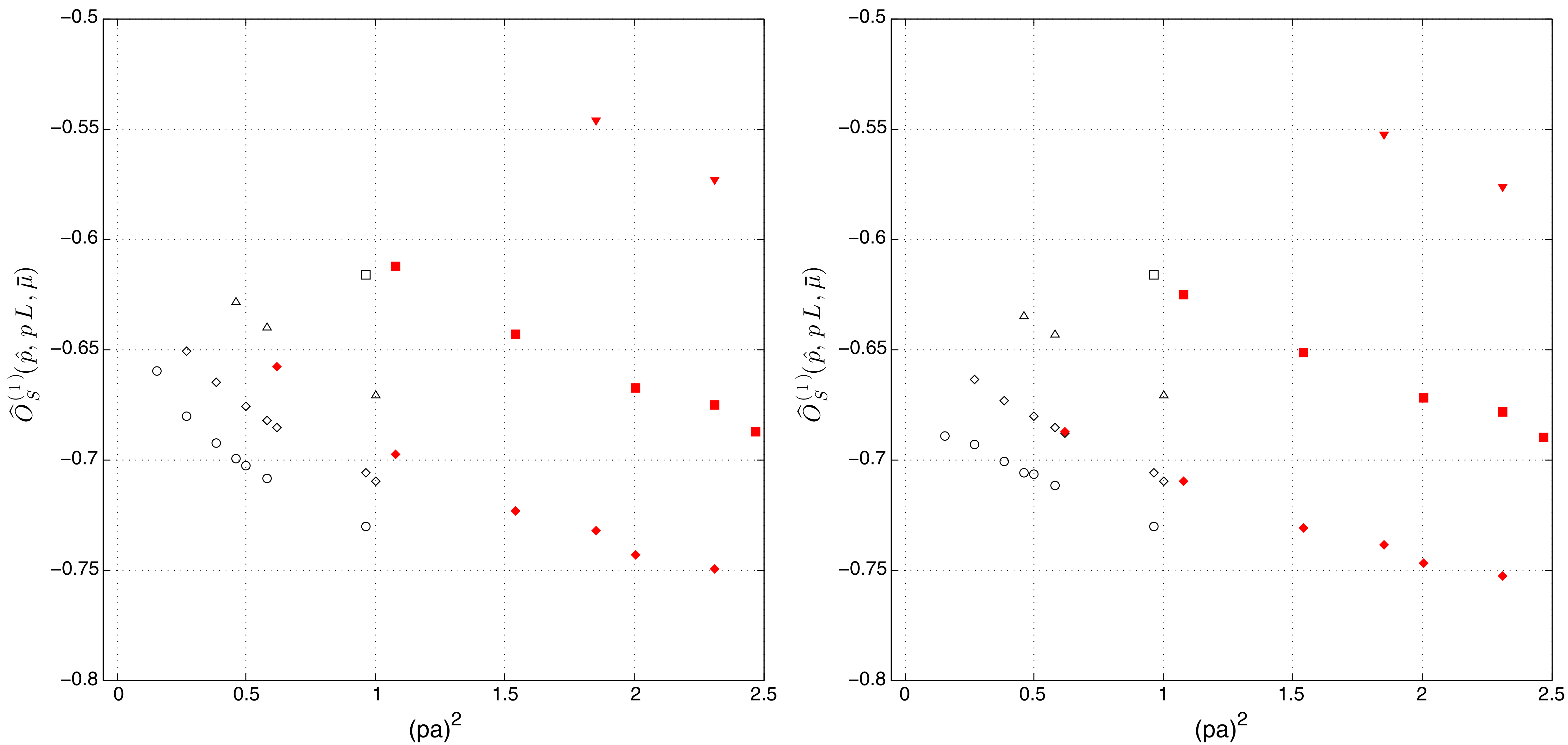
$$Z_{O_\Gamma}(\mu = p, \beta^{-1})|_{\text{finite part}} = \lim_{\substack{a\to 0\\ L\to\infty}} \frac{\widehat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu})}{\hat{O}_\Gamma(\hat{p}, pL)}|_{\text{log subtr}}$$

where the limits are encoded in expansions, e.g.

$$\widehat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu})|_{\text{log subtr}} = c_1^{(0)} + c_2^{(0)} \sum_\nu \hat{p}_\nu^2 + c_3^{(0)} \frac{\sum_\nu \hat{p}_\nu^4}{\sum_\nu \hat{p}_\nu^2} + c_1^{(1)} p_{\bar{\mu}}^2 + \Delta\widehat{\Sigma}_\gamma(pL) + \mathcal{O}(a^4)$$

and finite size effects come from

$$\widehat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu}) \equiv \widehat{\Sigma}_\gamma(\hat{p}, \infty, \bar{\mu}) + \Delta\widehat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu}) \qquad \Delta\widehat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu}) \sim \Delta\widehat{\Sigma}_\gamma(pL)$$



Three-loop computations of RI-MOM renormalization constants (*) Parma group 2007, 2013, 2014

(*) for different glue action