

RESURGENCE, BPS COUNTING AND KNOT INVARIANTS

Marcos Mariño
University of Geneva

The hidden structure of perturbation theory

Perturbation theory in a small parameter remains one of the most fruitful approaches to quantum theories.

Thanks to the theory of resurgence, we have learned that a perturbative series encodes much more information than originally thought. By an appropriate decoding of such a series, one can find e.g. additional sectors of the path integral.

This decoding involves beautiful and challenging mathematics, in which one passes from formal power series to analytic complex functions and their underlying geometry.

As I will show in this talk, the resurgent structure of perturbative series involves in some cases a hidden **integrality structure** which can be physically interpreted in terms of counting BPS states in a “dual” theory.

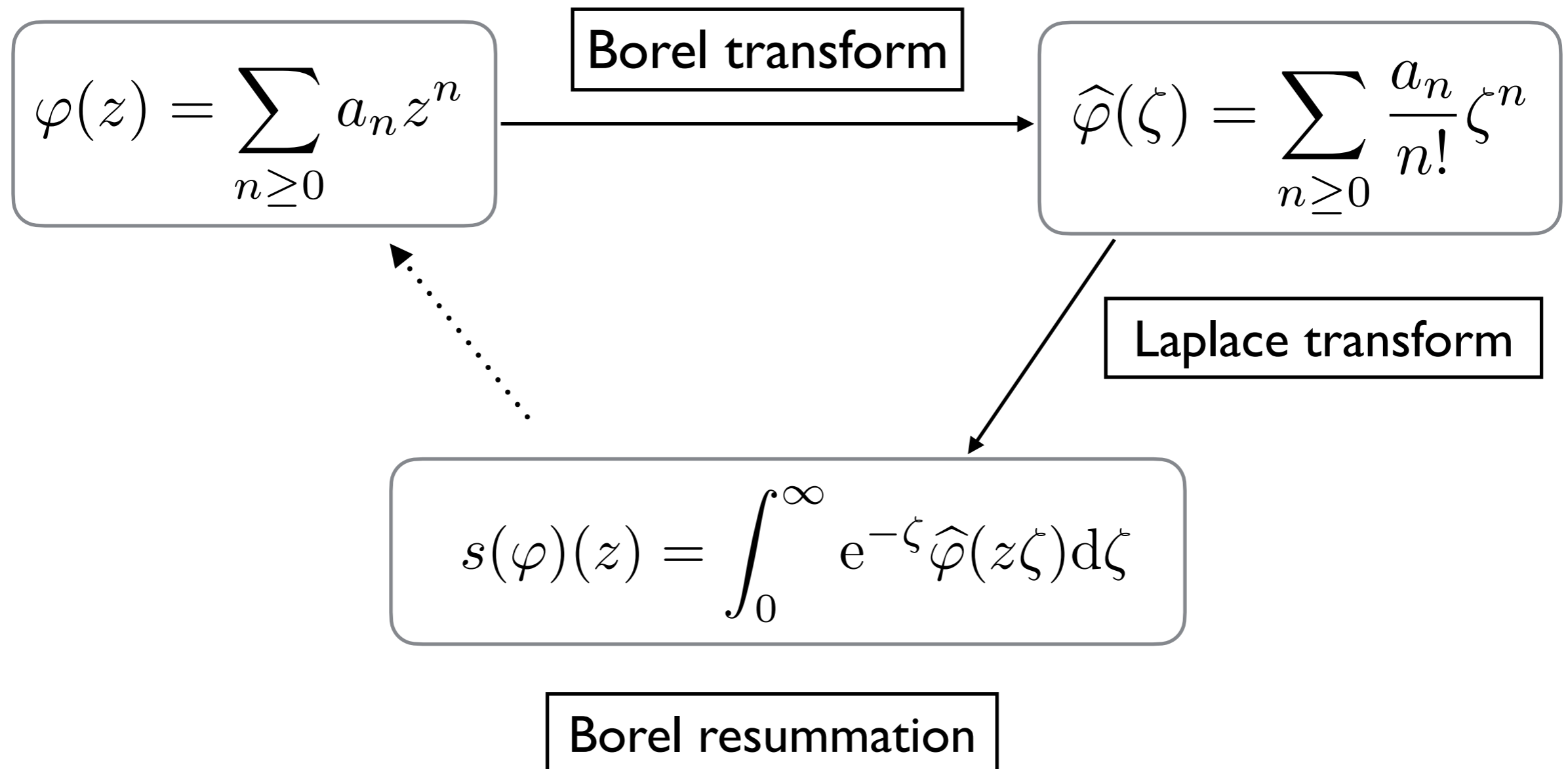
Two very interesting examples of such a situation are complex Chern-Simons theory and topological string theory. In this talk I will focus on the first one.

This is based on 2007.10190 with Stavros Garoufalidis and Jie Gu.

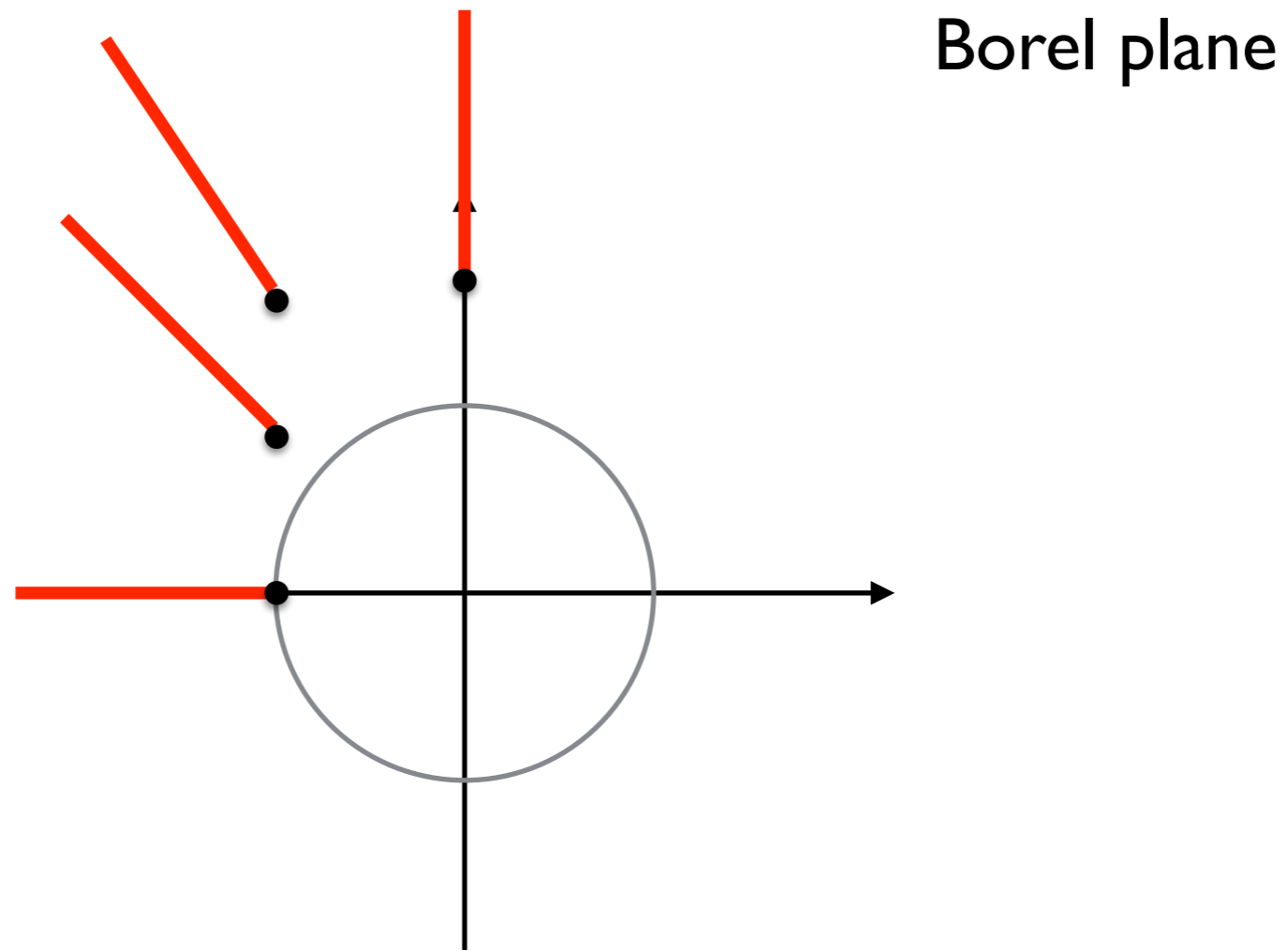


The Borel triangle

The Borel method is a systematic (and traditional) way of making sense of factorially divergent formal power series



The Borel transform $\hat{\varphi}(\zeta)$ is analytic at the origin. Very often it can be analytically continued to the complex plane, displaying a set of **singularities** (poles, branch cuts)



The expansion of the Borel transform around each singularity leads to new formal power series. For so-called simple resurgent functions, we have only log singularities:

$$\hat{\varphi}(\zeta) = -S_\omega \hat{\varphi}_\omega(\zeta - \zeta_\omega) \frac{\log(\zeta - \zeta_\omega)}{2\pi i} + \dots$$

Stokes constant

$$\varphi_\omega(z) = \sum_{n \geq 0} a_{n,\omega} z^n$$

These new perturbative series are typically associated to new sectors of the theory (e.g. expansions around different saddle points)

We can repeat the same analysis for all these new functions.

We conclude that, starting from a perturbative series, we generate **a set of formal power series**, corresponding to the different sectors of the path integral, and **a matrix of Stokes constants**

$$\varphi_{\omega}(z) \quad S_{\omega\omega'} \quad \begin{matrix} \omega \\ \text{labels the sectors} \end{matrix}$$

Extracting all this information in a given physical theory is not easy, to put it mildly.

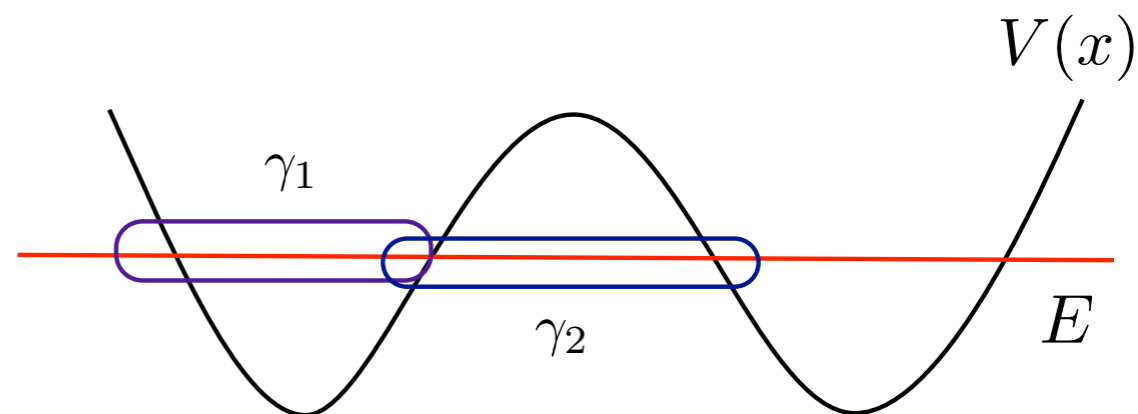
Stokes constants are often complicated numbers. However, in some cases they turn out to be **integers**. In simple examples this has a geometric interpretation in terms of intersection of integration paths (cf. Kontsevich's talk)

Exact WKB and BPS states

In the exact WKB method, the formal power series are (exponentiated) quantum periods, or Voros symbols

“WKB curve”

$$\Sigma(x, p) = H(x, p) - E = 0$$



quantum
periods

$$\Pi_\gamma(\hbar) = \oint_\gamma p(x, \hbar) dx = \sum_{n \geq 0} \Pi_\gamma^{(n)} \hbar^{2n}$$

Voros
symbols

$$\varphi_\gamma(\hbar) = e^{\Pi_\gamma(\hbar)}$$

The Stokes constants for this problem are integers that can be computed with the exact WKB method [Voros, Ecalle, Delabaere-Dillinger-Pham...] and they depend on the moduli of the curve.

Suppose that we can regard the WKB curve Σ as a **Seiberg-Witten curve** for a supersymmetric theory. One consequence of the work of Gaiotto-Moore-Neitzke (GMN) is that Stokes constants compute BPS multiplicities:

$$S_{\gamma\gamma'} = \langle \gamma, \gamma' \rangle \Omega(\gamma')$$

Complex Chern-Simons theory

A rich source of perturbative series (and beautiful mathematics!) is complex CS theory on a 3-manifold M .

When M is the complement of a hyperbolic knot K , it has been argued that the partition function of the theory can be reduced to a finite-dimensional **“state integral”** [Kashaev, Hikami, Dimofte et al., Andersen-Kashaev]

$$Z_K(\tau) = \int e^{-W(\mathbf{x};\tau)/\tau} d\mathbf{x} \quad \tau \in \mathbb{C} \setminus (-\infty, 0]$$

The building block of the integrand is Faddeev’s quantum dilogarithm. We will focus on $SL(2, \mathbb{C})$

Saddle-points correspond to flat complex connections σ on M .
 The expansions around these connections have the form

$$\exp\left(\frac{V(\sigma) + i\mathcal{C}(\sigma)}{2\pi}\right) \varphi_\sigma(\tau)$$

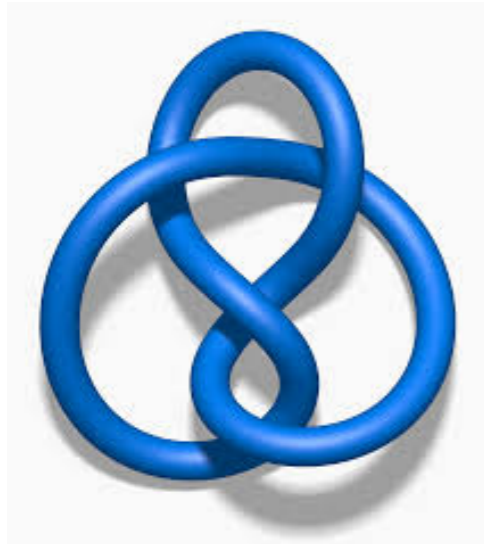
$$\varphi_\sigma(\tau) = \sum_{n \geq 0} a_n^\sigma \tau^n \quad \text{factorially divergent series!}$$

$$a_n^\sigma \sim n!$$

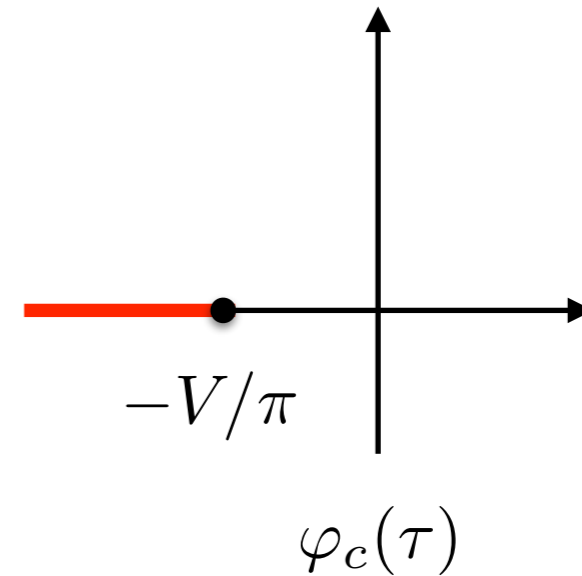
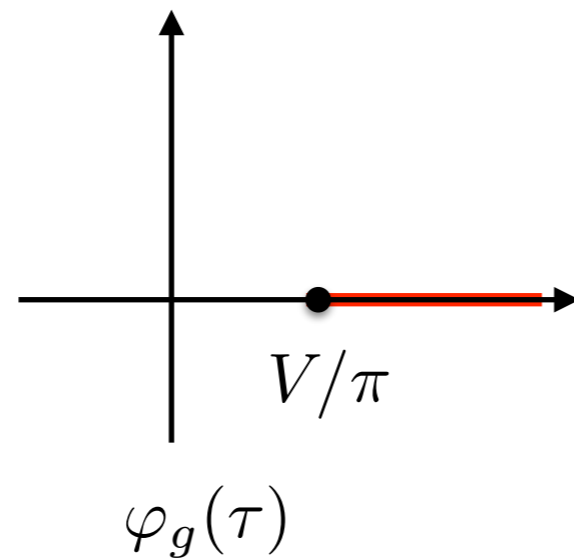
Among these connections there is always the “geometric connection” g (corresponding to the geodesically complete hyperbolic metric on M), and its conjugate c , with

$$V(g, c) = \pm V \quad \text{hyperbolic volume}$$

These saddle points lead to “classical” singularities
in the Borel plane.



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$$\varphi_g\left(\frac{\tau}{2\pi i}\right) = \frac{1}{3^{1/4}} \left(1 + \frac{11\tau}{72\sqrt{-3}} + \frac{697\tau^2}{2(72\sqrt{-3})^2} + \frac{724351\tau^3}{30(72\sqrt{-3})^3} + \dots \right)$$

=all-orders

Kashaev invariant

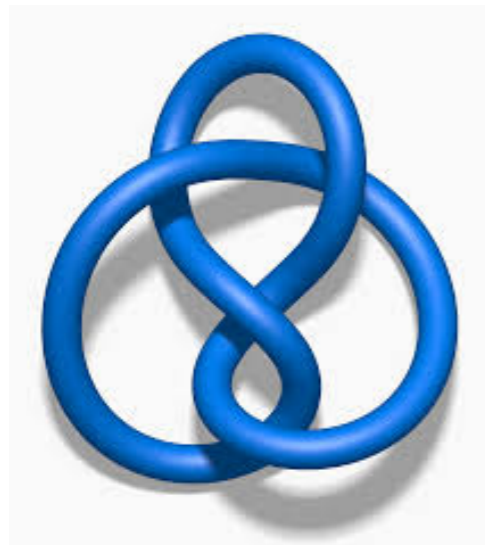
$$\varphi_c(\tau) = \varphi_g(-\tau)$$

These “classical” singularities and their Stokes constants (a finite number) were analyzed in e.g. [Gukov-M.M.-Putrov, Gang-Hatsuda, Garoufalidis-Zagier]

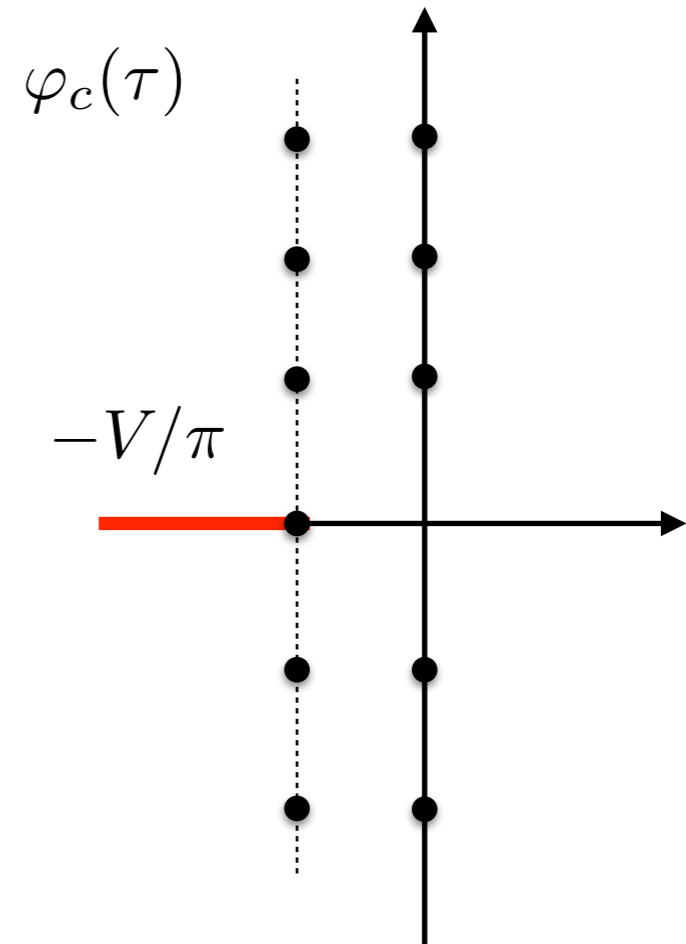
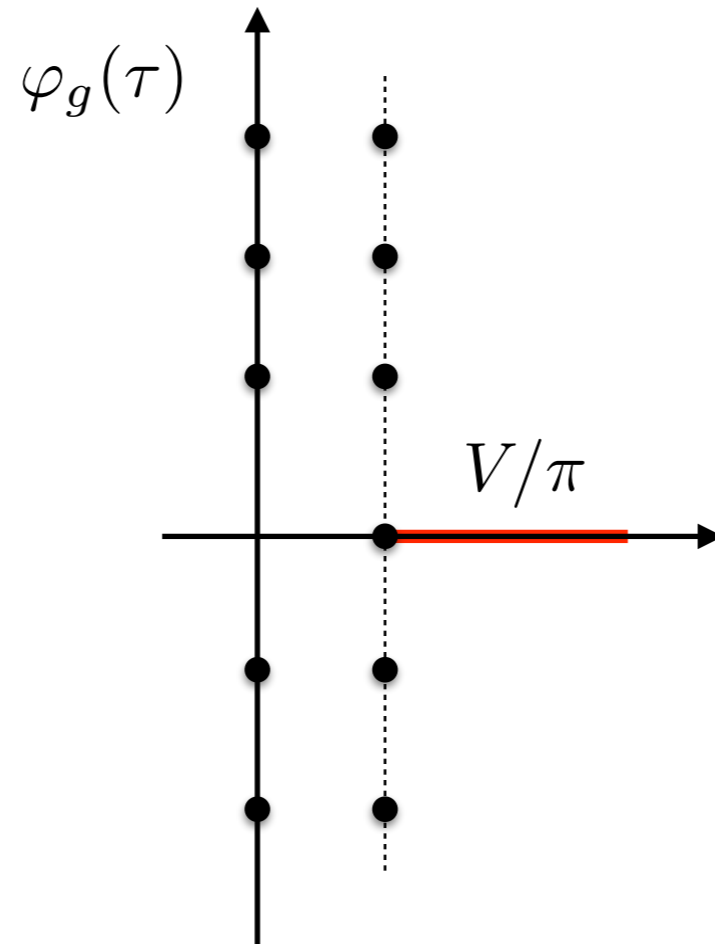
It turns however that, due to multivaluedness of the CS action and of the state integral potential, there are **infinite towers of additional singularities** [Garoufalidis, Witten, Gukov-M.M.-Putrov,...], corresponding to the “actions”

$$\frac{V(\sigma) + i\mathcal{C}(\sigma)}{2\pi} + 2\pi in \quad n \in \mathbb{Z}$$

The actual picture is rather



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Similar infinite towers of singularities appear in other contexts,
like topological string theory [Pasquetti-Schiappa, ...]

These towers correspond to tiny non-perturbative corrections
of order \tilde{q}^n

$$\tilde{q} = \exp\left(-\frac{2\pi i}{\tau}\right)$$

We have, in principle, infinitely many Stokes constants, labeled
by a pair of flat connections and a pair of integers

$$\omega = (\sigma, n) \qquad S_{\omega\omega'} = S_{\sigma,\sigma';n,n'}$$

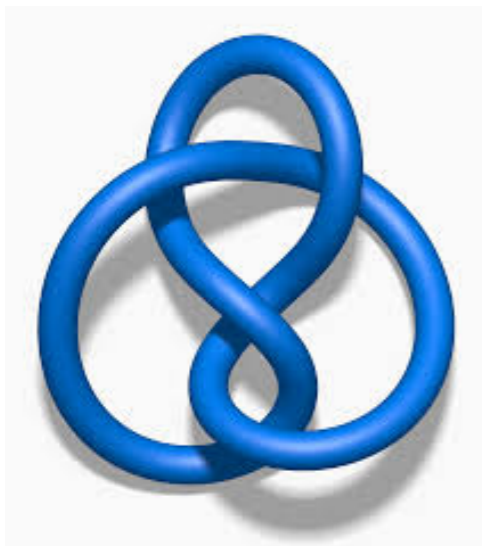
One might think that the towers of singularities are somewhat
“trivial” since they come from multivaluedness. Indeed, the
different formal power series have a simple multiplicative
structure

$$\varphi_{\sigma,n}(\tau) = \varphi_{\sigma}(\tau)\tilde{q}^n$$

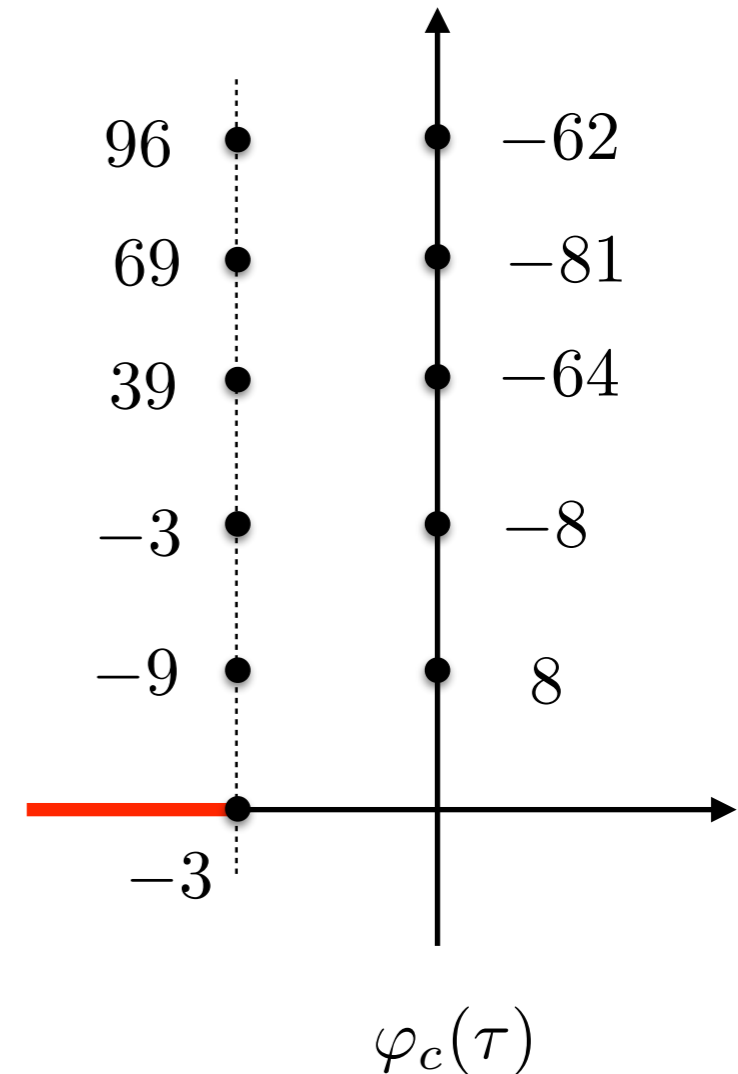
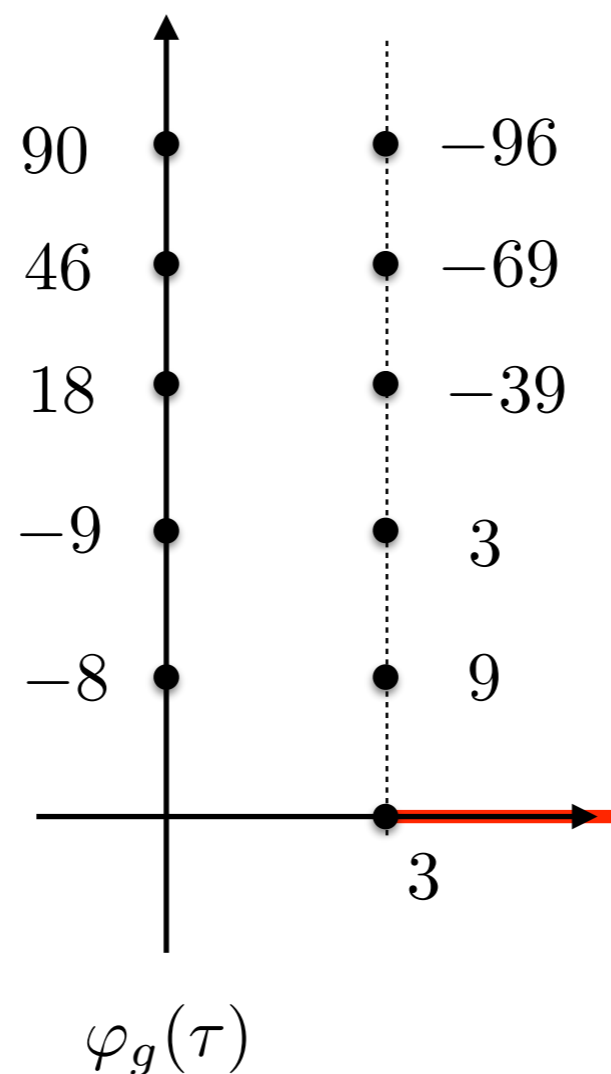
The Stokes constants depend then on a single integer n

$$S_{\sigma, \sigma'; n} = S_{\sigma, \sigma'; 0, n}$$

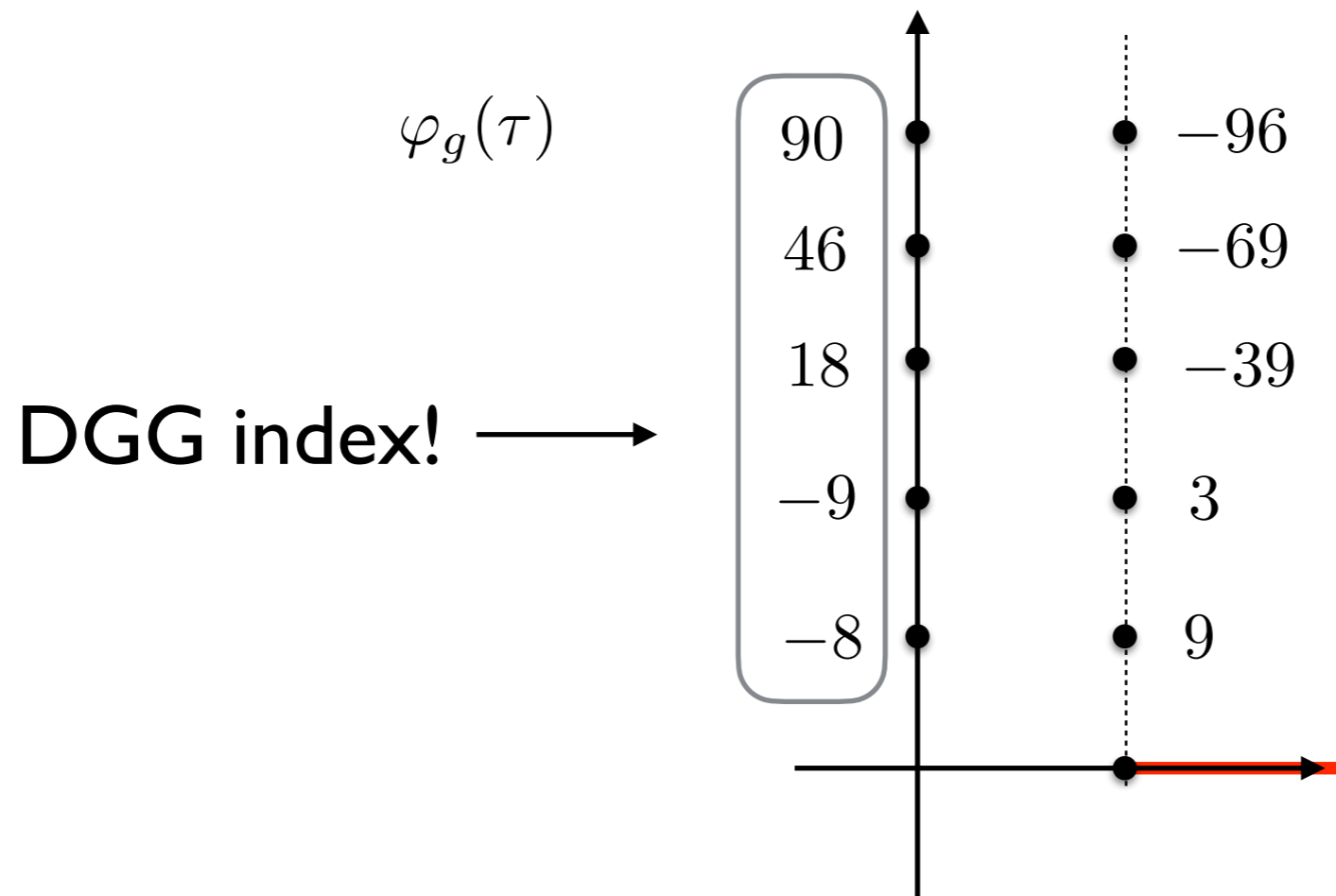
and explicit computations show that they provide
a highly non-trivial collection of integer invariants of the knot!



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Can we find a systematic description of these integers?



$$\mathrm{Tr}_{\mathcal{H}_{m=0}} (-1)^F q^{R/2+j_3} = 1 - 8q - 9q^2 + 18q^3 + 46q^4 + \dots$$

This counts BPS states in a 3d SCFT “dual” to the hyperbolic knot [Dimofte-Gaiotto-Gukov]

A more precise description involves the q -series appearing in the “block decomposition” of the state integral [Garoufalidis-Kashaev, Garoufalidis-Zagier] and generalizations thereof.

Consider the linear q -difference equation:

$$y_{m+1}(q) + y_{m-1}(q) - (2 - q^m)y_m(q) = 0$$

Then, the “Stokes q -series”

$$\mathcal{S}_{\sigma\sigma'}^{\pm}(q) = \sum_{n \geq 1} S_{\sigma,\sigma';n} q^{\pm n}$$

are given by explicit bilinear expressions in fundamental solutions of the above equation

Explicit formulae

$$g_\lambda(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)+n\lambda}}{(q)_n^2}$$

$$G_\lambda(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)+n\lambda}}{(q)_n^2} \left(2\lambda + E_1(q) + 2 \sum_{j=1}^n \frac{1+q^j}{1-q^j} \right)$$

$$g_+(q) = g_0(q) \qquad g_-(q) = g_1(q) + 2g_{-1}(q)$$

$$G_+(q) = G_0(q) \qquad G_-(q) = -G_1(q) - 2G_{-1}(q)$$

$$\mathcal{S}^+(q) = \begin{pmatrix} g_+ G_+ - 1 & 1 - g_- G_+ \\ 1 - g_+ G_- & g_- G_- - 1 \end{pmatrix}$$

We have obtained similar explicit results for the 5_2 knot.

General conjectures

We conjecture that, for any hyperbolic knot:

1. $\mathcal{S}_{gg}^+(q) = \text{DGG index} - 1$

2. All the Stokes q -series are bilinear expressions in solutions to a linear difference equation, which is in turn determined by the “block decomposition” of the state integral.

Related results have been recently announced by Kontsevich

“Stokes q -series” were previously calculated for $SU(2)$ CS on some Seifert spaces in [Costin-Garoufalidis, Gukov-M.M.-Putrov]

Conclusions and open questions

Resurgence can be used to extract precious information from perturbative series. In some cases, it leads to non-trivial integer invariants. This is a new route to integrality, different from previous ones (like radial asymptotics of q -series).

Our results determine (at least conjecturally) the **complete** resurgent structure of complex Chern-Simons theory for hyperbolic knots, and indicate a close relationship between Stokes constants and BPS counting in the “dual” 3d theory.

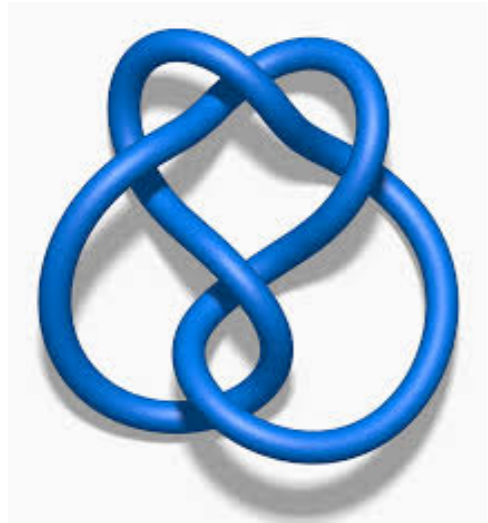
The same structures appear as well in topological string theory on toric CY manifolds (work in progress), but it is not clear what is the enumerative meaning of the resulting integers

Many open questions:

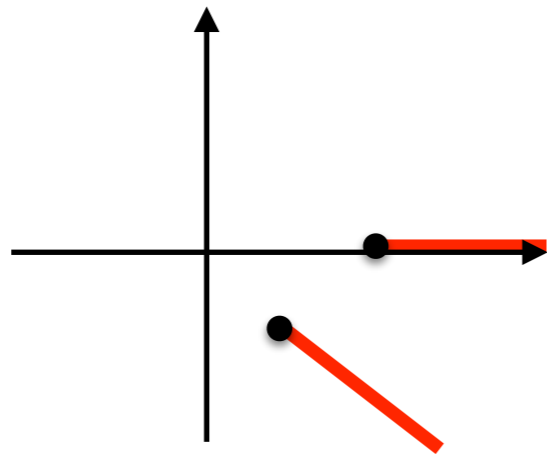
- 1) What happens if we turn on deformations of the hyperbolic structure? Do we find then a clearer relation to the A -polynomial and the AJ conjecture?
- 2) Can we obtain these results from a WKB analysis of the A -polynomial, a la GMN?
- 3) Can we reformulate the state integral invariants in terms of a Riemann-Hilbert problem?
- 4) Can we prove our conjectures or justify them physically?

Thank you for your attention!

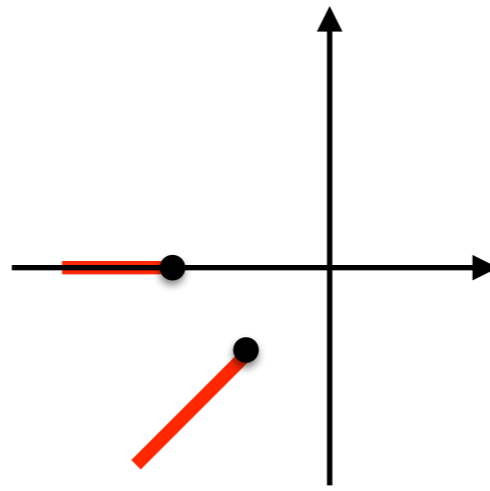




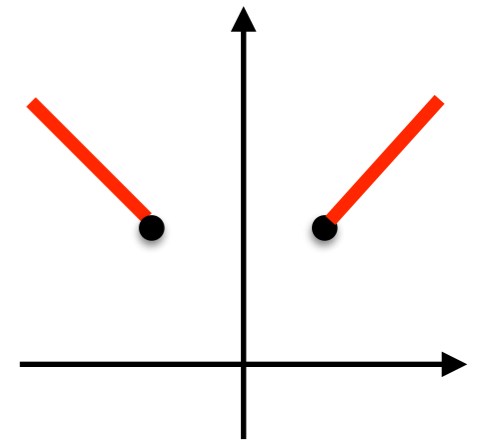
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$\varphi_g(\tau)$



$\varphi_c(\tau)$



$\varphi_r(\tau)$