The Sandwich Model

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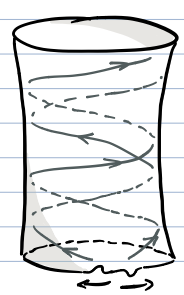
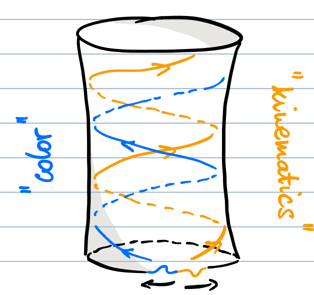
with Lorenz Eberhardt and Edward Witten

Motivation: Deeper understanding of loop momenta and double copy on the string worldsheet



string propagating in space-time



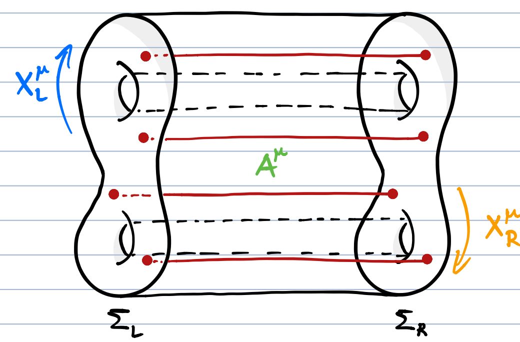
split into left + right movers + zero modes

How to do it at the level of the S-matrix?

→ Kawai-Lewellen-Tye '86 : tree-level factorization

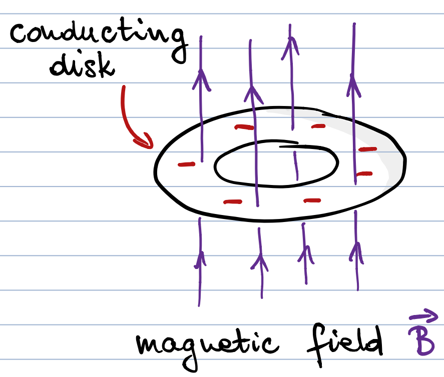
→ D'Hoker-Phong '88 : chiral splitting

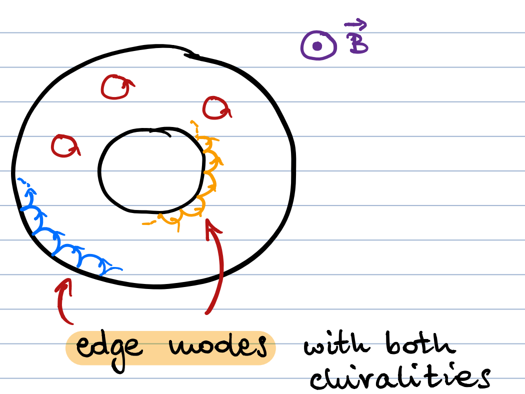
This talk: Physical interpretation of holomorphic factorization using the eponymous sandwich model:



Any suggestions for further applications are welcome!

But first, a little insight from condensed matter (Hall effect)



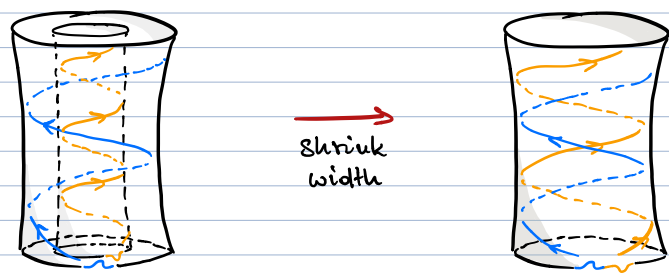


[Laughlin, Halperin, ... '80s]

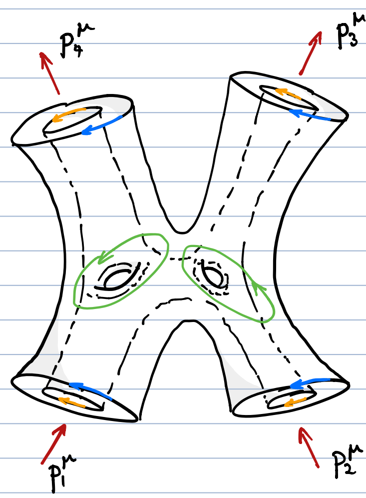
In the low-energy can be described by Chern-Simons theory:

$$S = \underbrace{\frac{k}{4\pi}\int_{\text{bulk}} \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)}_{\text{topological}} + \underbrace{\int_{\text{bdry}} \cdots}_{\text{all the dynamics}}$$

The width of the disk doesn't matter:

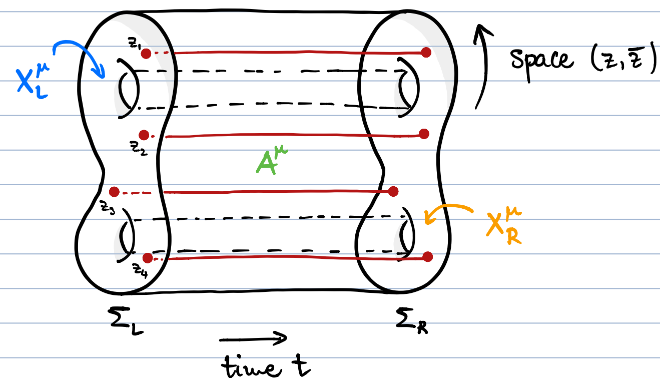


Could this lead to a theory of "topological membranes" with Chern-Simons worldvolume? [Kogan, Carlip, ... '80s]



We instead take it as a motivation for the following model on $M = \Sigma_g \times [0,1]$:

genus-g Riemann surface



Fields: $A^\mu = A_z^\mu + A_{\bar{z}}^\mu + A_t^\mu \in \Omega^1(M, \mathbb{R}^{1,d-1})$,

$X_L^\mu \in \Omega^0(\Sigma_L, \mathbb{R}^{1,d-1})$,

$X_R^\mu \in \Omega^0(\Sigma_R, \mathbb{R}^{1,d-1})$.

non-compact gauge group

Action: $$S_{cs} = \frac{k}{4\pi}\int_M \underbrace{A\,dA}_{\text{Abelian CS}} + \underbrace{\frac{4\pi}{k}\sum_{i=1}^{n} p_i A\,\delta^2(z-z_i)}_{\text{coupling to Wilson lines}}.$$

→ coupling constant not quantized: $k \notin \mathbb{Z}$

→ short-hand notation: $A\,dA = A^\mu \wedge dA_\mu$

→ no braiding of Wilson lines: $p_i A = p_{i,\mu} A_t^\mu$

Not well-defined in the presence of boundaries. Add:

$$S_L = \frac{k}{4\pi}\int_{\Sigma_L} \overbrace{(A_z - \partial X_L)A_{\bar z} + X_L(\partial A_{\bar z}}^{X_L \text{ couples chirally to } A} - \overbrace{\partial\bar\partial X_L}^{\text{kinetic term}}) + \underbrace{\frac{4\pi}{k}\sum_{i=1}^{n} p_i X_L\, \delta^2(z-z_i)}_{\text{coupling to Wilson lines}} .$$

$$S_R = \frac{k}{4\pi}\int_{\Sigma_R} A_z(\overbrace{A_{\bar z} + \bar\partial X_R}^{\text{opposite chirality}}) + X_R(\bar\partial A_z - \partial\bar\partial X_R) + \frac{4\pi}{k}\sum_{i=1}^{n} p_i X_R\, \delta^2(z-z_i).$$

Gauge transformation:

$$A^\mu \longrightarrow A^\mu + d\varepsilon^\mu,$$
$$X_L^\mu \longrightarrow X_L^\mu + \varepsilon^\mu,$$
$$X_R^\mu \longrightarrow X_R^\mu - \varepsilon^\mu.$$

Claim: The sandwich model computes scattering amplitudes.

$$\mathcal{Z} = \int \frac{\mathcal{D}A\, \mathcal{D}X_L\, \mathcal{D}X_R}{\mathrm{vol}\ \mathbb{R}^{1,d-1}}\, e^{i(S_{CS} + S_L + S_R)}.$$

Fix the temporal gauge : $A_t^\mu = 0$.

Equations of motion in the bulk :

→ $F^\mu_{z\bar{z}} = -\frac{2\pi}{k} \sum_{i=1}^{n} p_i^\mu \, \delta^2(z-z_i)$ ← Gauss constraint

→ $\partial_t A^\mu_z = \partial_t A^\mu_{\bar{z}} = 0$ ← gauge field is topological

And on the boundaries :

→ $A^\mu_z = \partial X^\mu_L$

→ $A^\mu_{\bar{z}} = -\bar{\partial} X^\mu_R$

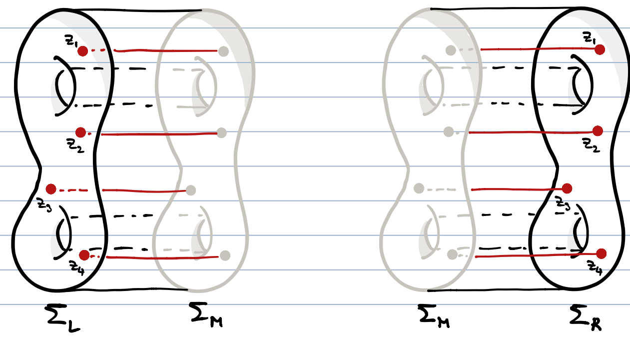
We can identify $\Sigma_L = \Sigma_R$ and compute the on-shell action:

$$\left. \cancelto{0}{S_{CS}} + S_L + S_R \right|_{EOM} = \frac{k}{4\pi} \int_\Sigma - \underbrace{(X_L + X_R)}_{X} \partial\bar{\partial} \underbrace{(X_L + X_R)}_{X} + \frac{4\pi}{k} \sum_{i=1}^{n} p_i \underbrace{(X_L + X_R)}_{X} \delta^2(z-z_i)$$

Therefore $\mathcal{Z}$ computes the genus-g contribution to the scattering amplitude of tachyons with $k = 2/\alpha'$!

(at fixed moduli z_i, Ω_{IJ})

Let's exploit a different point of view on the same problem:

$\Psi_L[A] =$  $= \Psi_R[A]$.

It gives an alternative representation of the amplitude:

$$\mathcal{Z} = \langle \Psi_L | \Psi_R \rangle = \int \frac{DA}{\mathrm{vol}\, \mathbb{R}^{1,d-1}} \Psi_L[A]\, \Psi_R[A] .$$

Geometric quantization of Chern-Simons theory

[Axelrod, S. Della Pietra, Witten '91, Witten '92]

→ Schrödinger equation:

$$\partial_t \Psi_L = 0 .$$

→ Gauss constraint:

$$\left[\partial_{\bar{z}} \left(\frac{\delta}{\delta A_{\bar{z}}} + i \frac{k}{4\pi} A_z \right) + \frac{k}{2\pi} F_{z\bar{z}} + i \sum_{i=1}^{n} p_i \, \delta^2(z - z_i) \right] \Psi_L = 0 .$$

→ Choice of holomorphic (Kähler) polarization:

$$\left(\frac{\delta}{\delta A_z} - i \frac{k}{4\pi} A_{\bar{z}} \right) \Psi_L = 0 .$$

Solved by

$$\Psi_L[A_z, A_{\bar z}] = e^{i\frac{k}{4\pi}\int_\Sigma A_z A_{\bar z}} \, \psi_L[A_{\bar z}]$$

and similarly

$$\Psi_R[A_z, A_{\bar z}] = e^{i\frac{k}{4\pi}\int_\Sigma A_z A_{\bar z}} \, \psi_R[A_z]$$

purely (anti) holomorphic

So that:

$$\langle \Psi_L | \Psi_R \rangle = \int \frac{DA}{\mathrm{vol}\ \mathbb{R}^{1,d-1}} \underbrace{e^{i\frac{k}{2\pi}\int_\Sigma A_z A_{\bar z}}}_{e^{-K}} \overbrace{\psi_L[A_{\bar z}]}^{\int DX_L \, e^{iS_L + k/2}} \underbrace{\psi_R[A_z]}_{\int DX_R \, e^{iS_R + k/2}} .$$

Kähler potential

To complete quantization of $H^1(\Sigma_g)$ we expand:

$$\rightarrow A^\mu_z = \frac{i}{2k}\left(\sum_{i=1}^n p_i^\mu \omega_{ir} + 2\pi i \sum_{I=1}^g \ell_I^\mu \omega_I \right) + \partial \eta$$

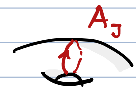
$$\rightarrow A^\mu_{\bar z} = -\frac{i}{2k}\left(\underbrace{\sum_{i=1}^n p_i^\mu \bar\omega_{ir}}_{\text{fixed by EOM}} \underbrace{- 2\pi i \sum_{I=1}^g \bar\ell_I^\mu \bar\omega_I}_{\substack{\text{unfixed DOF} \\ \sim \text{ loop momenta}}} \right) + \underbrace{\bar\partial \eta}_{\text{gauge fix } \eta = 0}$$

Loop momenta measure the gauge field along A-cycles:

$$\ell_I^\mu = -\frac{k}{\pi} \oint_{A_I} A_z^\mu , \qquad \bar\ell_I^\mu = -\frac{k}{\pi} \oint_{A_I} A_{\bar z}^\mu .$$

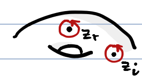
Quick reminder about Abelian differentials:

→ 1^{st} kind :



$$\oint_{A_J} \omega_I = \delta_{IJ} .$$

→ 3^{rd} kind :



$$\oint_{z_j=0} \omega_{ir} = \delta_{jr} - \delta_{ij} .$$

reference point z_r

Similarly,

$$X_L^\mu = \frac{i}{2k}\left(\sum_{i=1}^{n} p_i^\mu \int^z \omega_{ir} + 2\pi i \sum_{I=1}^{g} q_I^\mu \int^z \omega_I \right),$$

$$X_R^\mu = \frac{i}{2k}\left(\underbrace{\sum_{i=1}^{n} p_i^\mu \overline{\int^z \omega_{ir}}}_{\text{fixed by EOM}} - \underbrace{2\pi i \sum_{I=1}^{g} \bar{q}_I^\mu \overline{\int^z \omega_I}}_{\text{fixed by single-valuedness of } X_L + X_R} \right).$$

with $$q_I^\mu = - \sum_{i,J} \left(\mathrm{Im}\, \Omega_{IJ}^{-1} \right) p_i^\mu \, \mathrm{Im} \int^{z_i} \omega_J$$

To summarize, the path integral simplifies to

$$\langle \Psi_L | \Psi_R \rangle \sim \int \prod_{I=1}^{g} d\ell_I \, d\bar{\ell}_I \; e^{-k} \; \Psi_L[\bar{\ell}_I] \, \Psi_R[\ell_I] .$$

The only thing left to do is to evaluate the integrals of Abelian differentials :

→ $$\int_\Sigma \omega_I \wedge \overline{\omega}_J = -2i \,\mathrm{Im}\, \Omega_{IJ} ,$$

→ $$\int_\Sigma \omega_{ir} \wedge \overline{\omega}_{js} = -4\pi i \,\mathrm{Re} \int_{z_i}^{z_r} \omega_{js}$$

→ $$\int_\Sigma \omega_I \wedge \overline{\omega}_{js} = 4\pi \,\mathrm{Im} \int_{z_j}^{z_s} \omega_I ,$$

→ $$\int_\Sigma \omega_{ir} \wedge \overline{\omega}_J = -4\pi \,\mathrm{Im} \int_{z_i}^{z_r} \omega_J .$$

This gives the final answer:

$$\langle \Psi_L | \Psi_R \rangle \sim \int \prod_{I=1}^{g} dl_I \, d\bar{l}_I \left| e^{\frac{1}{4k}\left(\sum_{i,j} p_i \cdot p_j \int_{z_i}^{z_r} \omega_{jr} + 4\pi i \sum_{i,J} p_i \cdot \mathrm{Re}\, l_J \int_{z_i}^{z_r} \omega_J - 2\pi i \sum_{I,J} l_I \cdot \bar{l}_J \, \Omega_{IJ} \right)} \right|^2 .$$

→ Since $l_I \cdot \bar{l}_J = \mathrm{Re}\, l_I \cdot \mathrm{Re}\, l_J + \mathrm{Im}\, l_I \cdot \mathrm{Im}\, l_J$, the imaginary parts of loop momenta decouple ⇒ chiral splitting.

→ Integrating out $\mathrm{Re}\, l_I$ as well recovers the usual non-holomorphically factorized representation.

↑ Let's scroll through the slides for a recap ↑

Possible extensions:

→ Coupling to gravity: making z_i, Ω_{IJ} dynamical

→ Superstrings: adding fermions / super Riemann surfaces

→ Vertex operators: different reps. on Wilson lines

→ Versions of the sandwich model reproducing ambitwistor / sectorized string ?

Thanks!