Similarity matching: a new framework for neural computation

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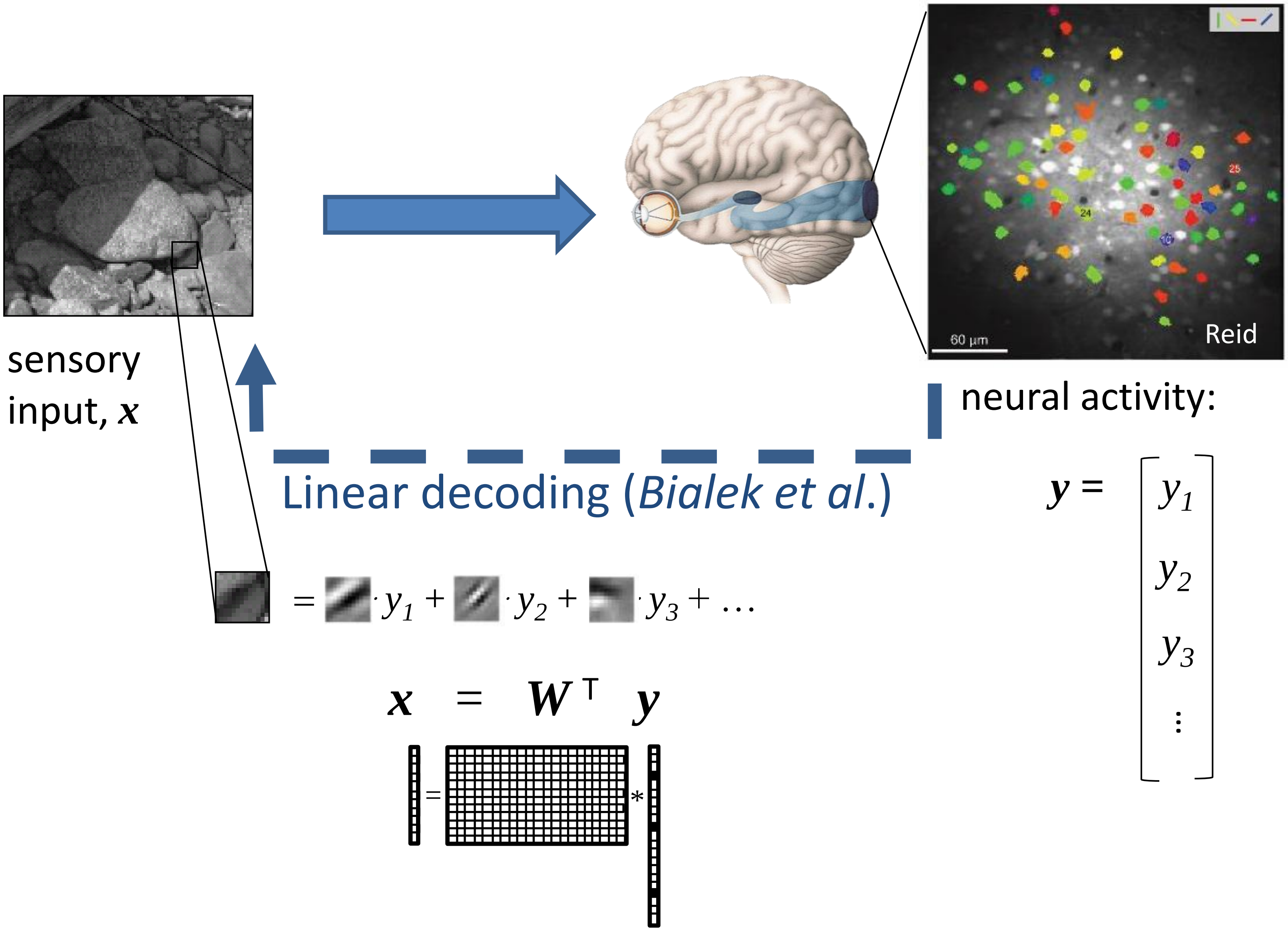
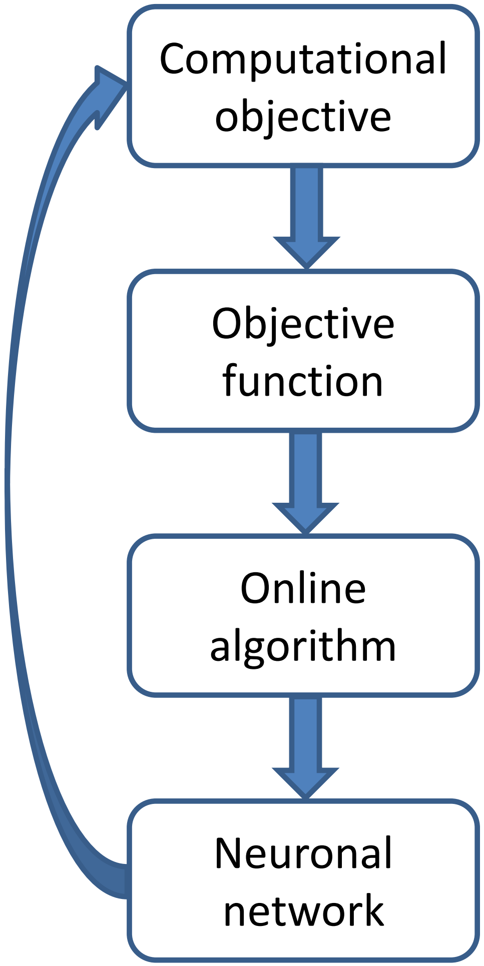
What does neural activity represent?

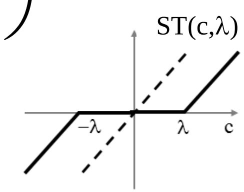


Olshausen and Field model (1996)



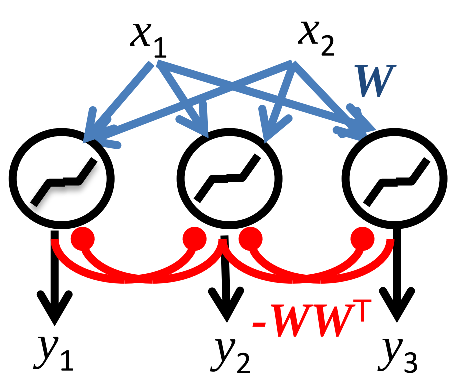
Sparse overcomplete representations, $\boldsymbol{y}_t$, of stimuli, $\boldsymbol{x}_t$

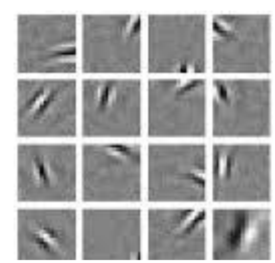
$$\min_{\boldsymbol{W}} \sum_t \min_{y_t} \left(\frac{1}{2} \left\| \boldsymbol{x}_t - \boldsymbol{W}^\top \boldsymbol{y}_t \right\|_2^2 + \lambda \left\| \boldsymbol{y}_t \right\|_1 \right)$$



$$\boldsymbol{y}_t \leftarrow \mathrm{ST}\left(\boldsymbol{W}\boldsymbol{x}_t - \boldsymbol{W}\boldsymbol{W}^\top \boldsymbol{y}_t, \lambda \right)$$

$$W_{i,j,t+1} \leftarrow W_{i,j,t} + y_{i,t} \left[x_{j,t} - \sum_k W_{k,j,t} y_{k,t} \right] \delta$$

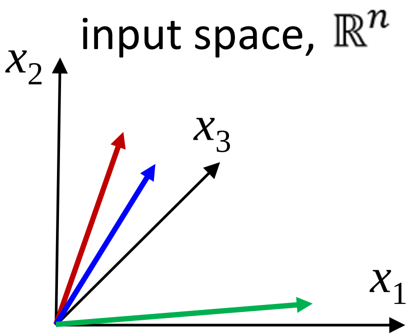


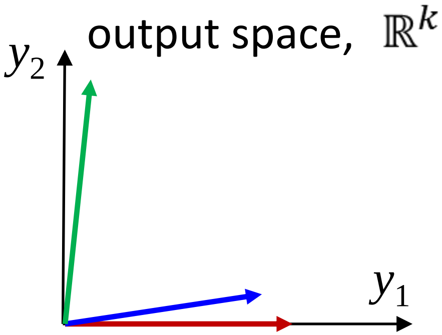


learned dictionary, $\boldsymbol{W}$

Alternative two-layer reciprocal circuit: Koulakov and Rinberg, 2011; Druckmann, Hu, Chklovskii, 2012

Similarity matching:
Similar inputs produce similar outputs





Quantify similarity by the scalar product

$$\boldsymbol{x}_1^\top \boldsymbol{x}_2 = \sum_{i=1}^{n} x_{i,1} x_{i,2} \qquad \boldsymbol{y}_1^\top \boldsymbol{y}_2 = \sum_{i=1}^{k} y_{i,1} y_{i,2}$$

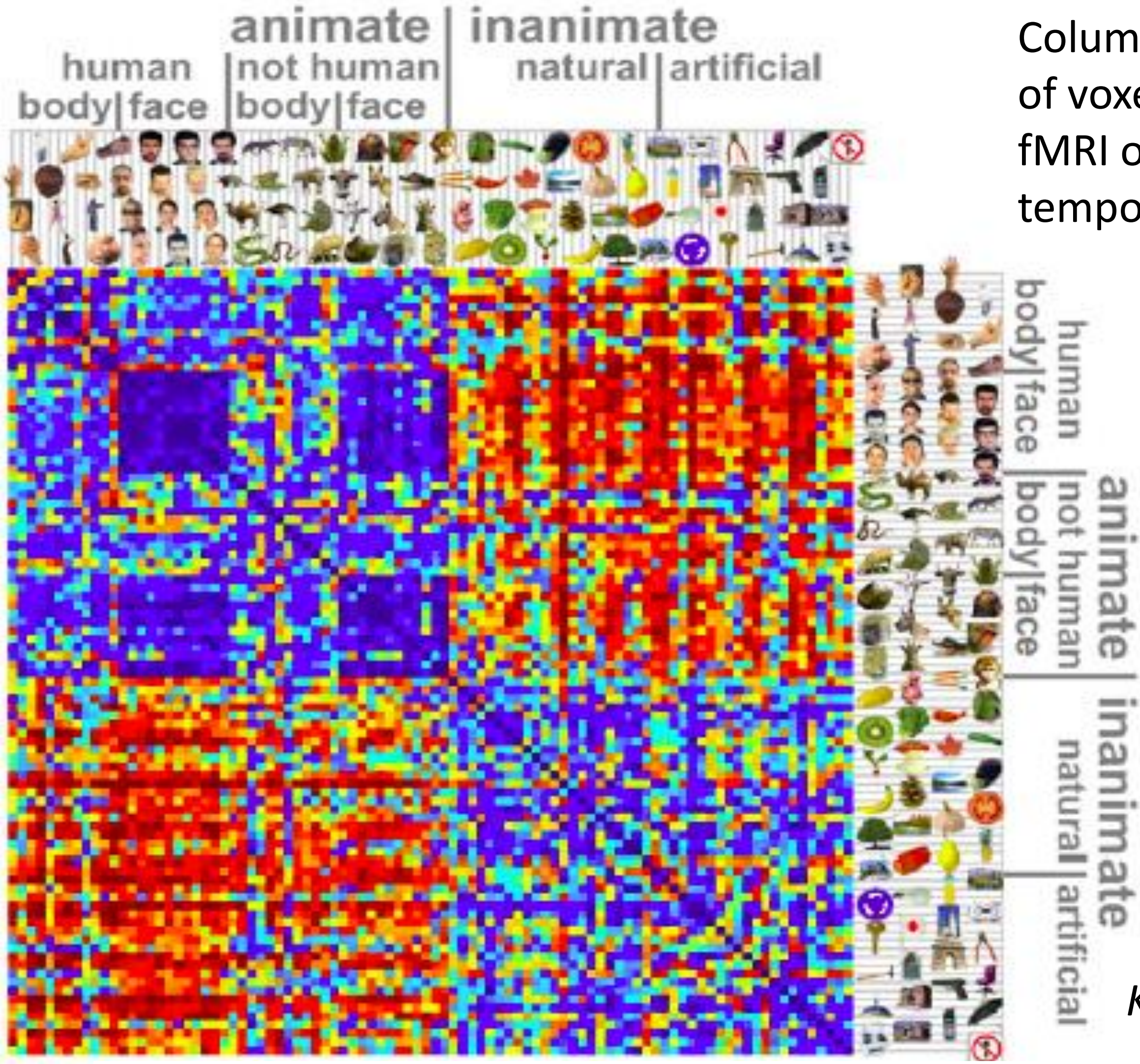
Matrix notation

$$\boldsymbol{X} = \left(\boldsymbol{x}_1 \ \cdots \ \boldsymbol{x}_t \right) \qquad \boldsymbol{Y} = \left(\boldsymbol{y}_1 \ \cdots \ \boldsymbol{y}_t \right)$$

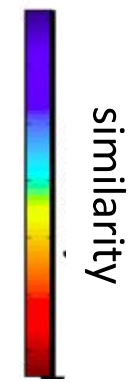
$$\boldsymbol{X}^\top \boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}_1^\top \boldsymbol{x}_1 & \boldsymbol{x}_1^\top \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_1^\top \boldsymbol{x}_t \\ \boldsymbol{x}_2^\top \boldsymbol{x}_1 & \boldsymbol{x}_2^\top \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_2^\top \boldsymbol{x}_t \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{x}_t^\top \boldsymbol{x}_1 & \boldsymbol{x}_t^\top \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_t^\top \boldsymbol{x}_t \end{pmatrix} \qquad \boldsymbol{Y}^\top \boldsymbol{Y} = \begin{pmatrix} \boldsymbol{y}_1^\top \boldsymbol{y}_1 & \boldsymbol{y}_1^\top \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_1^\top \boldsymbol{y}_t \\ \boldsymbol{y}_2^\top \boldsymbol{y}_1 & \boldsymbol{y}_2^\top \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_2^\top \boldsymbol{y}_t \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{y}_t^\top \boldsymbol{y}_1 & \boldsymbol{y}_t^\top \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_t^\top \boldsymbol{y}_t \end{pmatrix}$$

$$\boxed{\boldsymbol{X}^\top \boldsymbol{X} \approx \boldsymbol{Y}^\top \boldsymbol{Y}}$$

Empirical similarity matrix, $\boldsymbol{Y}^{\top}\boldsymbol{Y}$



Columns of $\boldsymbol{Y}$ are vectors of voxel activation from fMRI of human inferior temporal cortex



Kriegeskorte et al

More similar sensory stimuli evoke more similar neuronal activity patterns

Objective function for similarity matching

$$\min_{\boldsymbol{Y}} \left\| \boldsymbol{X}^\top \boldsymbol{X} - \boldsymbol{Y}^\top \boldsymbol{Y} \right\|_F^2$$

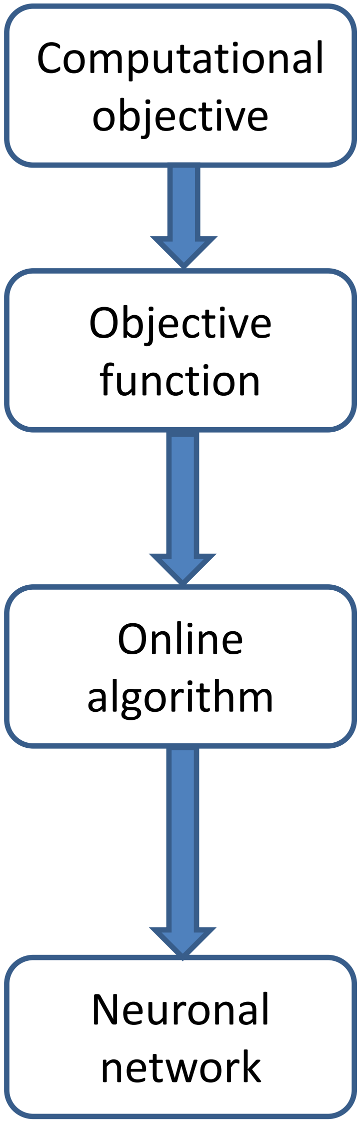
$$\min_{\substack{\boldsymbol{Y} \\ k<n}} \left\| \boldsymbol{X}^\top \boldsymbol{X} - \boldsymbol{Y}^\top \boldsymbol{Y} \right\|_F^2$$

principal subspace projection
MDS strain cost, PCA-like
Pehlevan, Hu, & Chklovskii (2015)

$$\min_{\substack{\boldsymbol{Y} \geq 0 \\ k<n}} \left\| \boldsymbol{X}^\top \boldsymbol{X} - \boldsymbol{Y}^\top \boldsymbol{Y} \right\|_F^2$$

soft clustering
symmetric NMF
Pehlevan & Chklovskii (2014)

Similarity matching with unconstrained output



Principal subspace projection

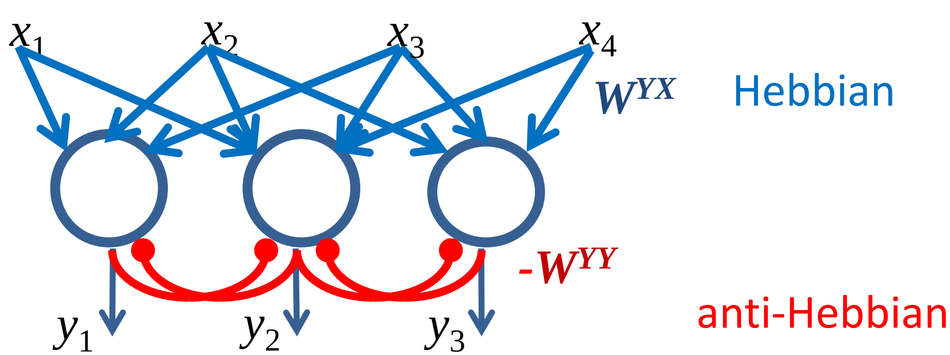
$$\min_{\boldsymbol{Y}} \left\| \boldsymbol{X}^{\top}\boldsymbol{X} - \boldsymbol{Y}^{\top}\boldsymbol{Y} \right\|_F^2$$

$$\boldsymbol{y}_t \leftarrow \arg\min_{\boldsymbol{y}_t} \left\| \boldsymbol{X}_t^{\top}\boldsymbol{X}_t - \boldsymbol{Y}_t^{\top}\boldsymbol{Y}_t \right\|_{\mathrm{F}}^2$$

$$W_{i,j,t}^{YX} = \frac{\sum_{\tau=1}^{t-1} y_{i,\tau} x_{j,\tau}}{\sum_{\tau=1}^{t-1} y_{i,\tau}^2}$$

$$\boldsymbol{y}_t \leftarrow \boldsymbol{W}_t^{YX}\boldsymbol{x}_t - \boldsymbol{W}_t^{YY}\boldsymbol{y}_t$$

$$W_{i,j,t+1}^{YX} \leftarrow W_{i,j,t}^{YX} + y_{i,t}\left(x_{j,t} - W_{i,j,t}^{YX} y_{i,t}\right) \Big/ \sum_{\tau} y_{i,\tau}^2$$



Online similarity matching learns principal subspace on synthetic data

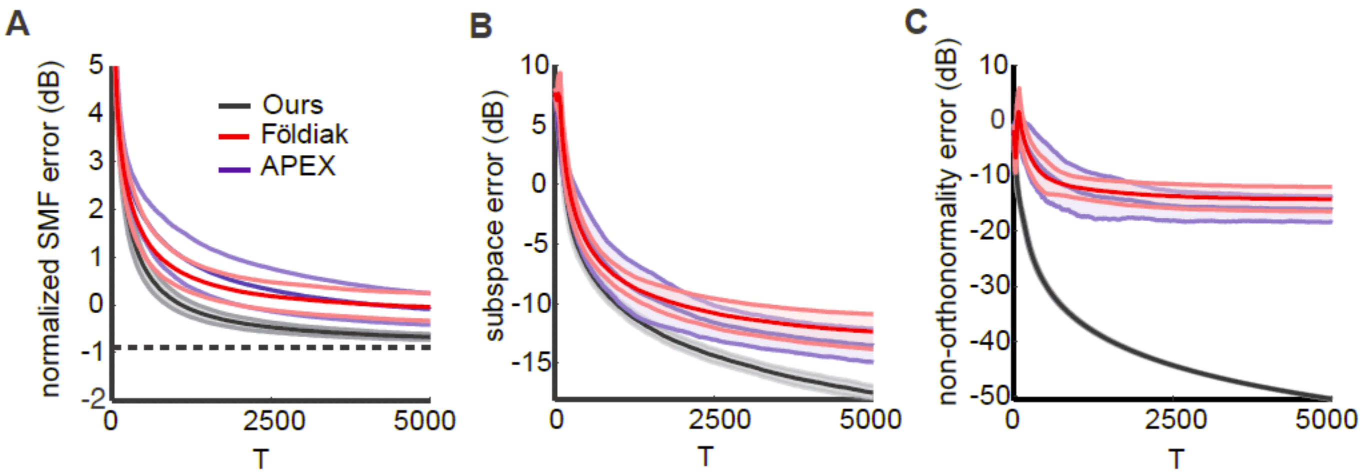


Fig. 2: Performance of the neural network compared with existing algorithms.

A. Normalized SMF error:

$$\frac{1}{T^2}\left\|\mathbf{X}^\top\mathbf{X}-\mathbf{Y}^\top\mathbf{Y}\right\|_F^2$$

B. Subspace error:

$$\mathbf{F}_T = (\boldsymbol{I}_k + \boldsymbol{W}_T^{YY})^{-1}\,\boldsymbol{W}_T^{YX}$$

$$\left\|\mathbf{F}_T^\top\mathbf{F}_T-\mathbf{V}^\top\mathbf{V}\right\|_F^2$$

C. Non-orthonormality error:

$$\left\|\mathbf{F}_T\mathbf{F}_T^\top-\mathbf{I}_m\right\|_F^2$$

Objective function for similarity matching

$$\min_{\boldsymbol{Y}} \left\| \boldsymbol{X}^\top \boldsymbol{X} - \boldsymbol{Y}^\top \boldsymbol{Y} \right\|_F^2$$

$$\min_{\substack{\boldsymbol{Y} \\ k<n}} \left\| \boldsymbol{X}^\top \boldsymbol{X} - \boldsymbol{Y}^\top \boldsymbol{Y} \right\|_F^2$$

principal subspace projection
MDS strain cost, PCA-like
Pehlevan, Hu, & Chklovskii (2015)

$$\min_{\substack{\boldsymbol{Y} \geq 0 \\ k<n}} \left\| \boldsymbol{X}^\top \boldsymbol{X} - \boldsymbol{Y}^\top \boldsymbol{Y} \right\|_F^2$$

soft clustering
symmetric NMF
Pehlevan & Chklovskii (2014)

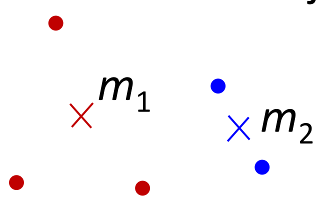
Similarity matching objective is a relaxation of *k*-means clustering

Ding et al, 2005

k-means objective:

$$\min_{\{k'\}} \sum_{k'=1}^{k} \sum_{t \in C_k} \left\| \boldsymbol{x}_t - \boldsymbol{m}_{k'} \right\|_2^2$$

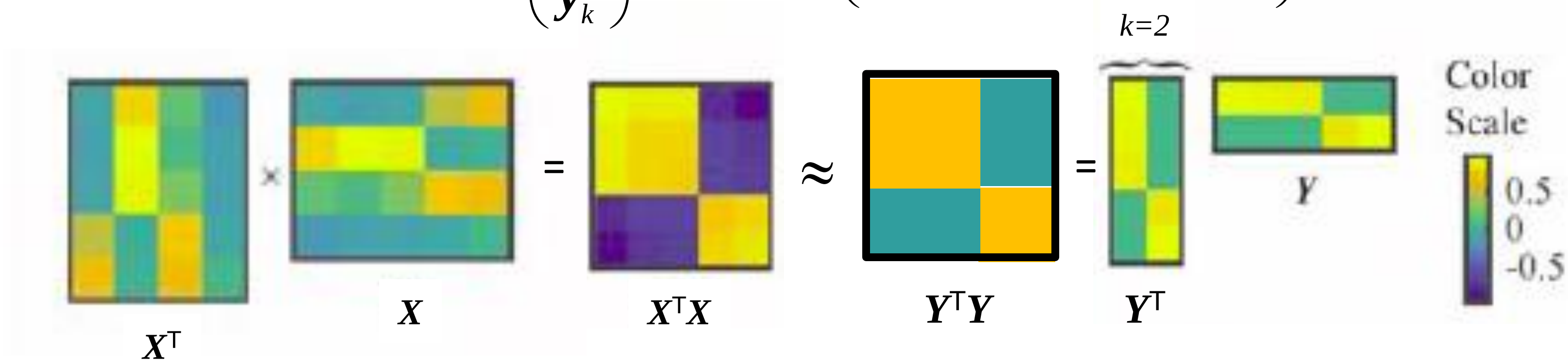
$$\boldsymbol{Y} \leftarrow \underset{\boldsymbol{Y} \geq 0}{\arg\min} \left\| \boldsymbol{X}^\top \boldsymbol{X} - \boldsymbol{Y}^\top \boldsymbol{Y} \right\|_F^2$$



$$\boldsymbol{m}_{k'} = \frac{1}{n_{k'}} \sum_{t \in C_{k'}} \boldsymbol{x}_t$$

Indicator function:

$$\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_k \end{pmatrix}, \boldsymbol{y}_{k'} = \frac{1}{n_{k'}^{1/2}} \Big(0, \cdots, 0, \overbrace{1, \cdots, 1}^{n_{k'}}, 0, \cdots, 0 \Big)$$



Similarity matching with nonnegative output

Computational objective

↓

Objective function

↓

Online algorithm

↓

Neuronal network

Soft clustering

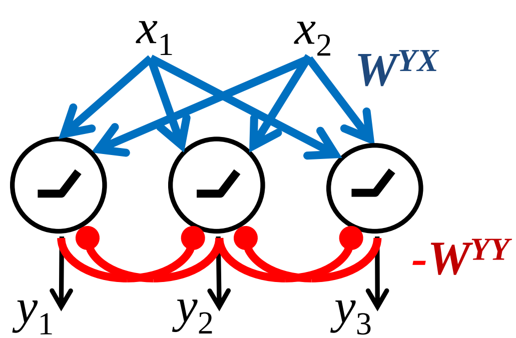
$$\min_{\boldsymbol{Y}\geq 0}\left\|\boldsymbol{X}^{\top}\boldsymbol{X}-\boldsymbol{Y}^{\top}\boldsymbol{Y}\right\|_F^2$$

$$\boldsymbol{y}_t \leftarrow \arg\min_{\boldsymbol{y}_t\geq 0}\left\|\boldsymbol{X}_t^{\top}\boldsymbol{X}_t-\boldsymbol{Y}_t^{\top}\boldsymbol{Y}_t\right\|_{\mathrm{F}}^2$$

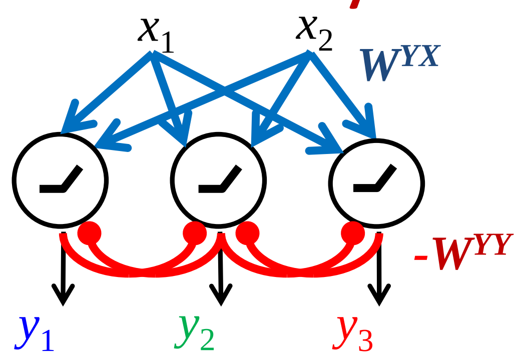
$$\boldsymbol{y}_t \leftarrow \max\left(\boldsymbol{W}_t^{YX}\boldsymbol{x}_t-\boldsymbol{W}_t^{YY}\boldsymbol{y}_t, 0\right)$$

$$W_{i,j,t}^{YX}=\frac{\sum_{\tau=1}^{t-1} y_{i,\tau}x_{j,\tau}}{\sum_{\tau=1}^{t-1} y_{i,\tau}^2}$$

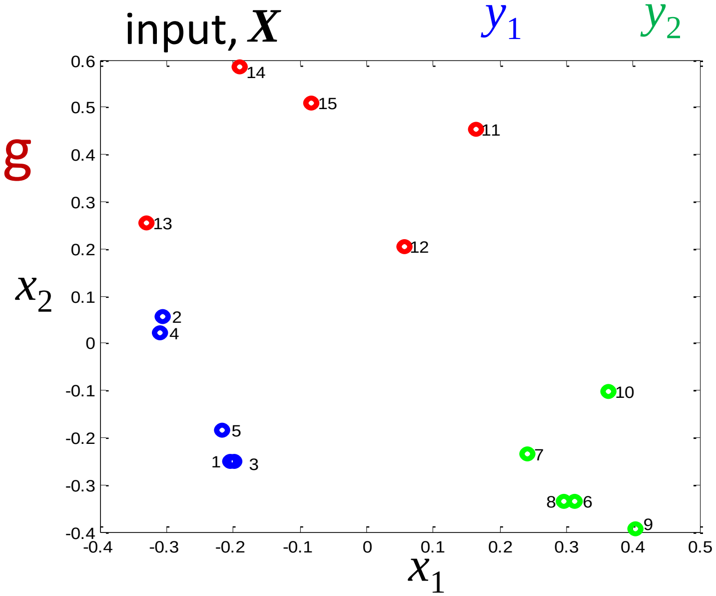
$$W_{i,j,t+1}^{YX}\leftarrow W_{i,j,t}^{YX}+y_{i,t}\left(x_{j,t}-W_{i,j,t}^{YX}y_{i,t}\right)\Big/\sum_{\tau}y_{i,\tau}^2$$

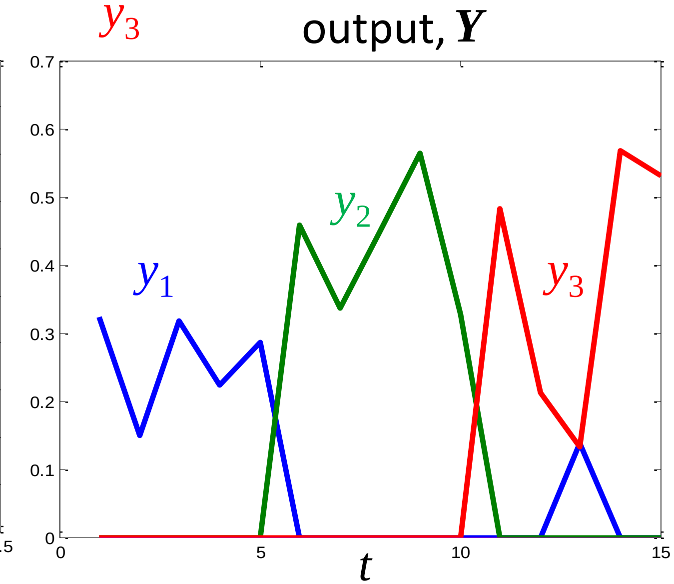


Online similarity matching

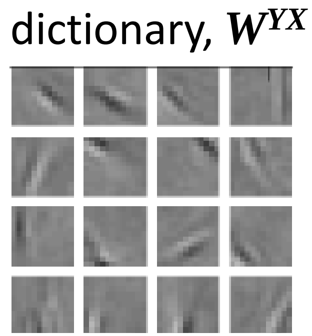


Soft clustering





Learning sparse representation



Similarity matching: a new framework for neural computation

- *Online similarity matching can be implemented by neural dynamics in a circuit with local learning rules*
- *Similarity matching with unconstrained output projects onto a principal subspace*
- *Similarity matching with nonnegative output clusters and extracts independent components*
- *Output dimensionality can adapt to the input*

Acknowledgements

Cengiz Pehlevan *Tao Hu* -> Texas A&M

Positions available in the Neuroscience Group
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