Noncommutative motives,

Thermodynamics and the zeros of zeta

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Work in progress

Prime Numbers

$\pi(n) =$ number of prime numbers $p \leq n$

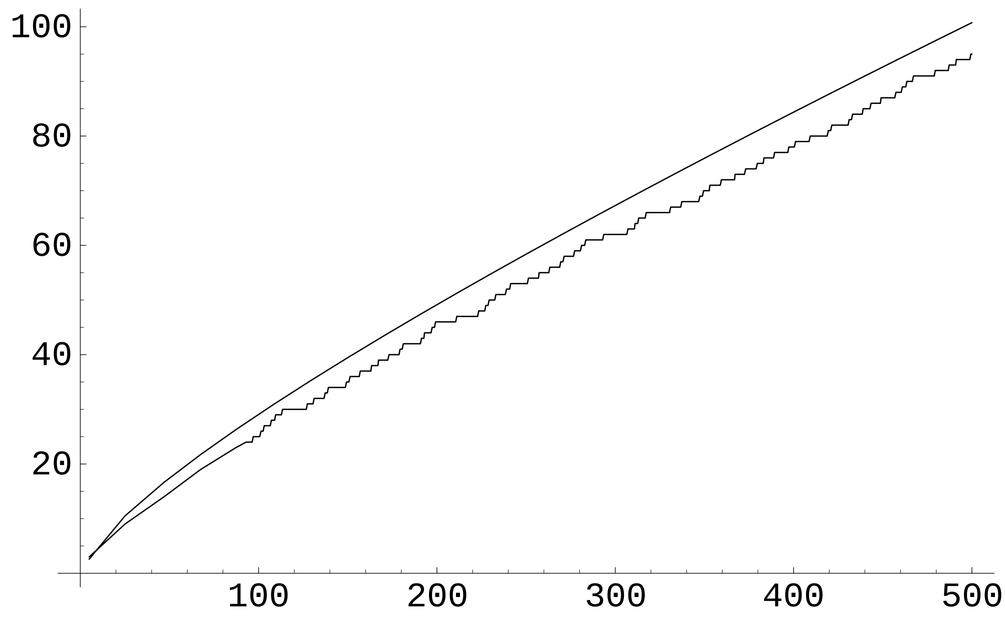
$$Li(x) = \int_0^x \frac{du}{\log(u)} \sim \sum (k-1)! \frac{x}{\log(x)^k}$$

$$\pi(x) = \int_0^x \frac{du}{\log(u)} + R(x)$$

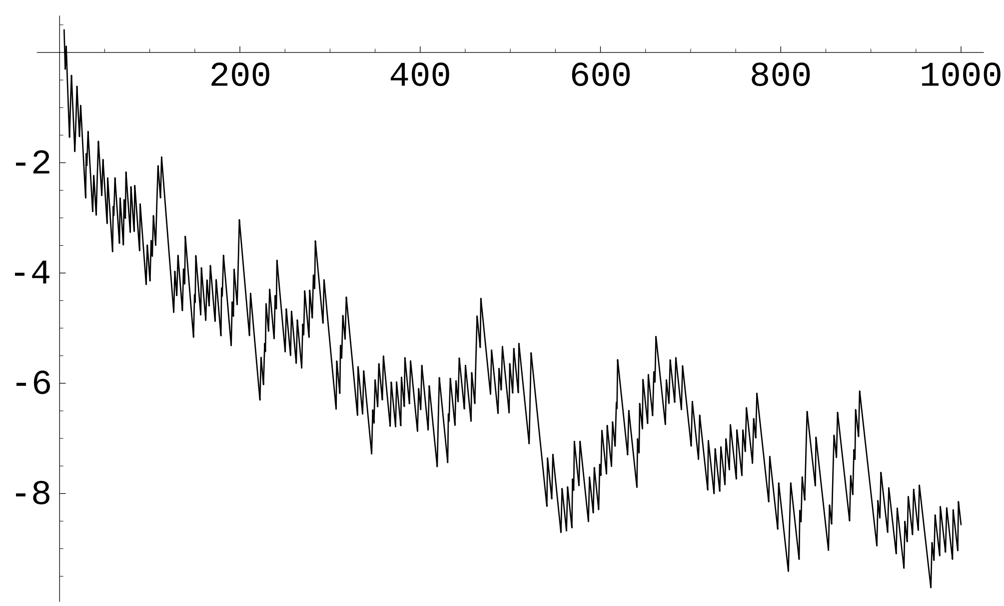
Riemann Conjecture :

$$R(x) = O(\sqrt{x} \log(x))$$

$$(\pi(n) = 2 + \sum_5^n \frac{e^{2\pi i \Gamma(k)/k} - 1}{e^{-2\pi i/k} - 1}, \quad \Gamma(k) = (k-1)!)$$



Graphs of $\pi(x)$ and Li(x)



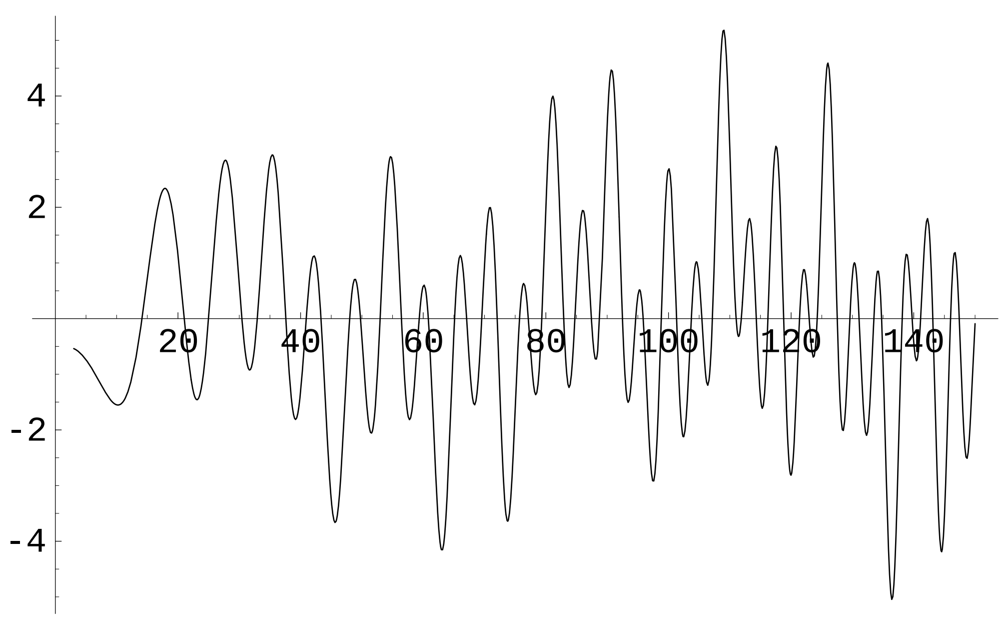
Graph of $\pi(x)-$ Li(x)

Zeta Function

$$\zeta(s) = \sum_{1}^{\infty} n^{-s} = \prod_{\mathcal{P}} (1 - p^{-s})^{-1}$$

$$\zeta_{\mathbb{Q}}(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$$

$$s \to 1 - s$$



Explicit Formula (Riemann)

$$\pi'(x) = Li(x) - \sum_{\rho} Li(x^{\rho})$$

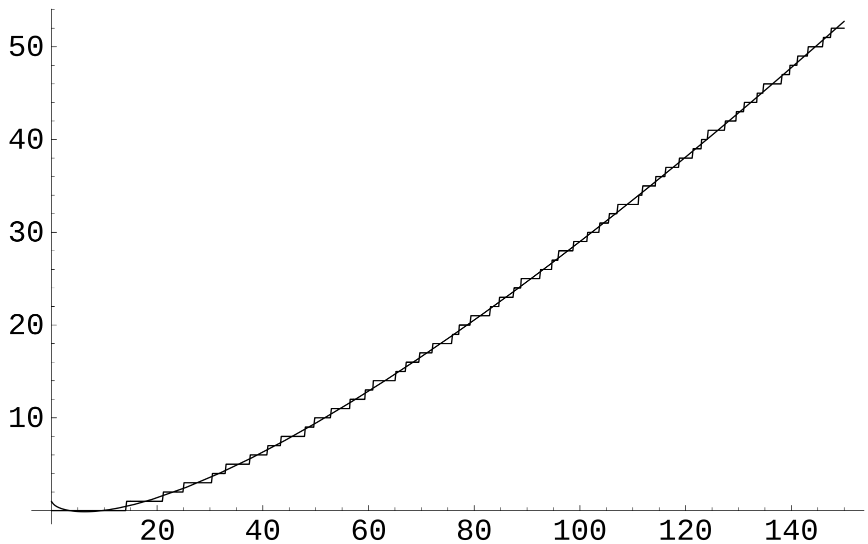
$$+ \int_{x}^{\infty} \frac{du}{u(u^2 - 1)\log u} - \log 2$$

$$\pi'(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots$$

Explicit Formula (Weil)

$$\widehat{h}(0) + \widehat{h}(1) - \sum_{\rho} \widehat{h}(\rho) = \sum_{v} \int_{K_v^*} \frac{h(u^{-1})}{|1 - u|} d^*u$$

Quantum Chaos → Riemann Flow ?



$$N(E) = \langle N(E) \rangle + N_{\mathrm{osc}}(E)$$

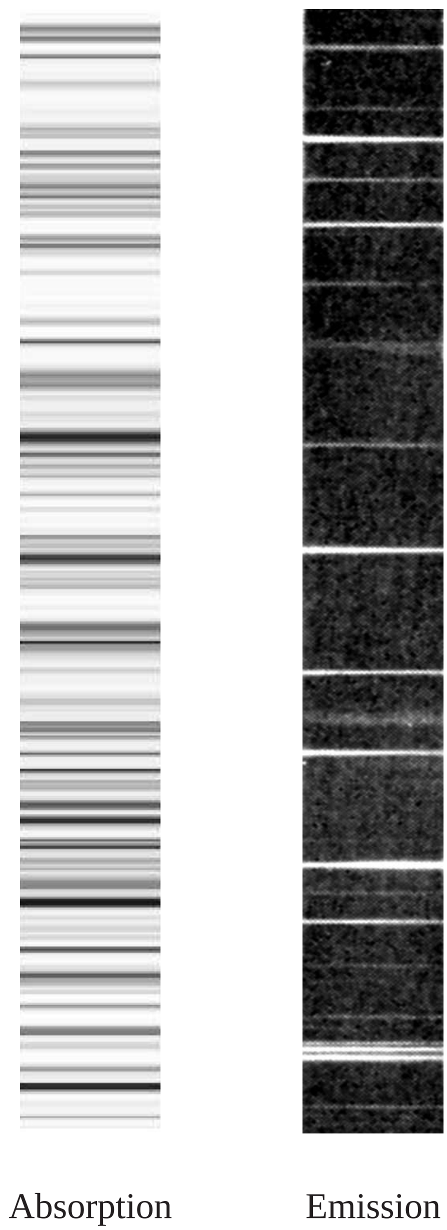
$$\langle N(E) \rangle = \frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right) + \frac{7}{8} + o(1)$$

Sign Problem :

$$N_{\text{osc}}(E) \sim \frac{-1}{\pi} \sum_{p} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{p^{m/2}} \sin\left(m\, E\, \log\, p\right)$$

$$N_{\text{osc}}(E) \sim \frac{1}{\pi} \sum_{\gamma_p} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{2\text{sh}\left(\frac{m\lambda_p}{2}\right)} \sin(m\, E\, T_{\gamma}^{\#})$$

Absorption Spectrum



The two kinds of Spectra

$\mathbb{Q}$-**Lattices** (ac + mm)

A $\mathbb{Q}$-lattice in $\mathbb{R}^n$ is a pair (Λ, ϕ), with Λ a lattice in $\mathbb{R}^n$, and

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda$$

a homomorphism of abelian groups.

Two $\mathbb{Q}$-lattices (Λ_1, ϕ_1) and (Λ_2, ϕ_2) are commensurable if the lattices are commensurable (*i.e.* $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$) and the maps agree modulo the sum of the lattices,

$$\phi_1 \equiv \phi_2 \mod \Lambda_1 + \Lambda_2.$$

$X_{\mathbb{Q}}$ = space of 1-dimensional $\mathbb{Q}$-lattices modulo commensurability.

Spectral realization

Idele class group $\widehat{\mathbb{Z}}^* \times \mathbb{R}_+^*$ acts on $L^2(X_{\mathbb{Q}})$ and **zeros of L-functions give the absorption spectrum** with non-critical zeros appearing as resonances.

$$\text{Trace}\,(R_\Lambda\, U(h)) = 2h(1)\log' \Lambda +$$

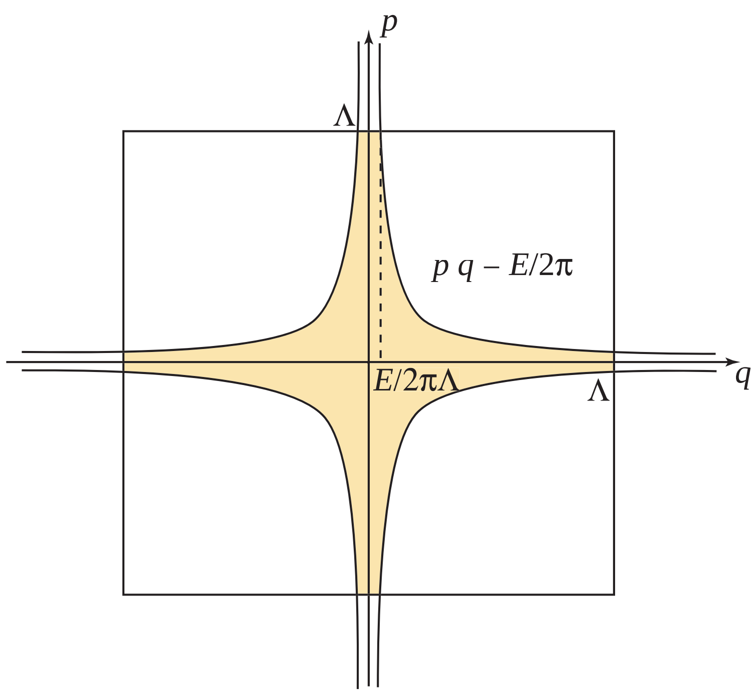
$$\sum_{v\in S} \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)$$

$\int'$ is the pairing with the distribution on k_v which agrees with $\frac{du}{|1-u|}$ for $u \neq 1$ and whose Fourier transform relative to α_v vanishes at 1.

Global Trace Formula $\Leftrightarrow$ RH

$\langle N(E)\rangle$ as symplectic volume $|h| \leq E$

$$h(q,p) = 2\pi q\, p$$



$$\mathrm{Vol}(B_+) = \frac{E}{2\pi} \times 2\log\Lambda - \frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right)$$

Global field of positive characteristic

k is the field of $\mathbb{F}_q$ valued functions on C.

$$\zeta_k(s) = \prod_{\Sigma_k} (1 - q^{-f(v)s})^{-1}$$

$f(v)$ is the degree of the place $v \in \Sigma_k$.

Functional Equation

$$q^{(g-1)(1-s)}\,\zeta_k(1-s) = q^{(g-1)s}\,\zeta_k(s)$$

where g is the genus of C.

Cohomology and Frobenius

$$\zeta_k(s) = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

where P is the caracteristic polynomial of the action of the **Frobenius** Fr^* in $H^1_{\mathrm{et}}(\bar{C}, \mathbb{Q}_\ell)$.

The analogue of the Riemann conjecture for global fields of characteristic p means that the eigenvalues of the action of Fr^* in H^1 *i.e.* the complex numbers λ_j of the factorization

$$P(T) = \prod (1 - \lambda_j T)$$

are of modulus $|\lambda_j| = q^{1/2}$.

Proved by Weil (1942) (case $g = 1$ by Hasse)

Frobenius in characteristic zero

(ac + cc +mm)

- **Thermodynamics of noncommutative spaces**

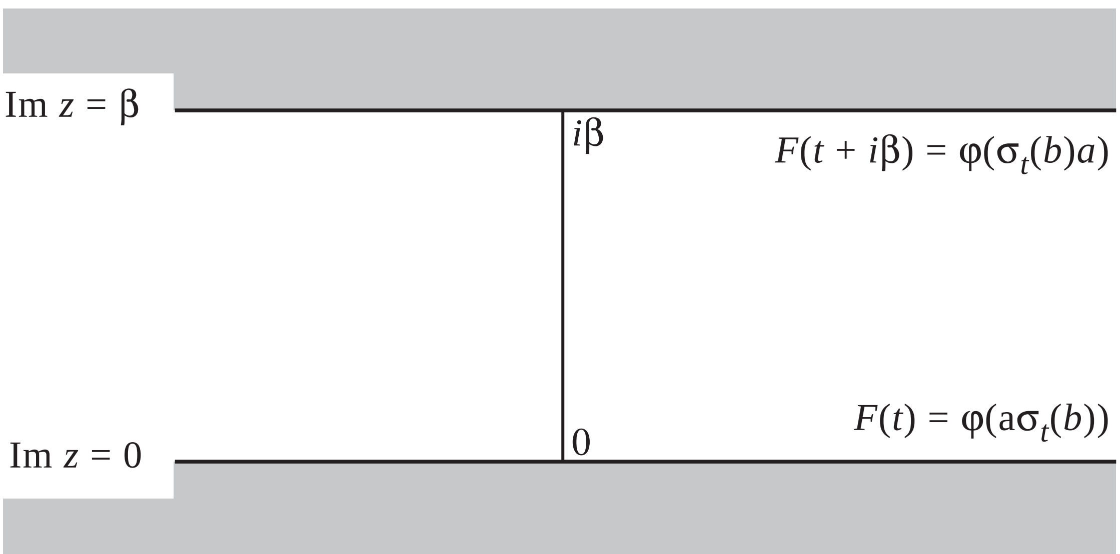
- **Category of Λ-modules = abelian category** (Λ = cyclic category)

- **Endomotives**

The KMS condition

$$\varphi(x^* x) \geq 0 \qquad \forall\, x \in \mathcal{A}\,,\ \varphi(1) = 1\,.$$

$$\sigma_t \in \mathrm{Aut}(\mathcal{A})$$



$$F_{x,y}(t) = \varphi(x\sigma_t(y))$$

$$F_{x,y}(t + i\beta) = \varphi(\sigma_t(y)x), \qquad \forall t \in \mathbb{R}.$$

Cooling :

$\mathcal{E}_\beta$ extremal KMS_β states, for $\beta > 1$

$$\rho : \mathcal{A} \rtimes_\sigma \mathbb{R} \to \mathcal{S}(\mathcal{E}_\beta \times \mathbb{R}_+^*) \otimes \mathcal{L}^1$$

Distillation :

Λ-module $D(\mathcal{A}, \varphi)$ given by the Cokernel of the cyclic morphism given by the composition of ρ with the trace $\text{Tr} : \mathcal{L}^1 \to \mathbb{C}$

Dual action :

Spectrum of the canonical action of $\mathbb{R}_+^*$ on the cyclic homology

$$HC_0(\,D(\mathcal{A}, \varphi))$$

Endomotives

A is an inductive limit of reduced finite dimensional commutative algebras over the field $\mathbb{K}$ and S is a semigroup of algebra endomorphisms

$$\rho : A \to A$$

$$\mathcal{A}_{\mathbb{K}} = A \rtimes S$$

Prototype Example :

Endomorphisms of an algebraic variety (group),

$$X_s = \{y \in Y : s(y) = *\}.$$

$$X_{sr} \ni y \mapsto r(y) \in X_s.$$

$$X = \varprojlim_s X_s$$

$$\xi_{su}(\rho_s(x)) = \xi_u(x)$$

Explicit Formula = Trace Formula (ac + rm + cc +mm)

$$\text{Trace}_{H^1}(h) = \widehat{h}(0) + \widehat{h}(1) - \sum_v \int_{K_v^*} \frac{h(u^{-1})}{|1-u|} d^*u$$

were the last term $\sum_v \int_{K_v^*} \frac{h(u^{-1})}{|1-u|} d^*u$ is the intersection number

$$Z(h) \bullet \Delta$$

$$\text{Trace}_{H^1}(h) = \widehat{h}(0) + \widehat{h}(1) - \Delta \bullet \Delta \, h(1)$$

$$- \sum_v \int_{(K_v^*, e_{K_v})} \frac{h(u^{-1})}{|1-u|} d^*u$$

Unramified extensions $K \to K \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$

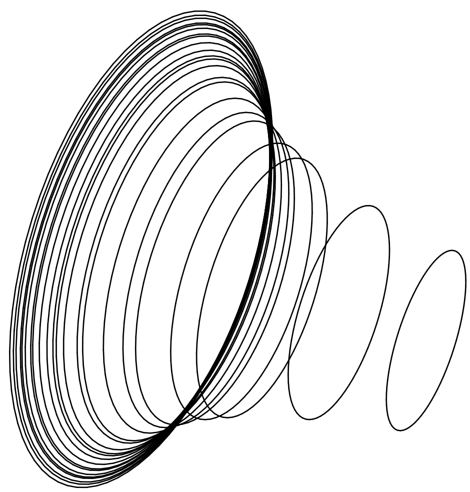
Analogue for $\mathbb{Q}$ of $K \to K \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$

Global field K	**Factor** M
$\mathrm{Mod}\, K \subset \mathbb{R}_+^*$	$\mathrm{Mod}\, M \subset \mathbb{R}_+^*$
$K \to K \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$	$M \to M \rtimes_{\sigma_T} \mathbb{Z}$
$K \to K \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$	$M \to M \rtimes_{\sigma} \mathbb{R}$
Points $C(\bar{\mathbb{F}}_q)$	$\Gamma \subset X_{\mathbb{Q}}$

The subspace $\Gamma_{\mathbb{Q}} \subset X_{\mathbb{Q}} \backslash C_{\mathbb{Q}}$

$$\Gamma_{\mathbb{Q}} = \cup_{\Sigma_{\mathbb{Q}}} C_{\mathbb{Q}}[v] \subset X_{\mathbb{Q}}$$

$$[v]_w = 1\,, \quad \forall w \neq v\,, \quad [v]_v = 0$$



$\ldots\ldots \mathrm{Log}\, P \ldots\ldots\ldots\ldots \quad \mathrm{Log}7\,, \quad \mathrm{Log}5\,, \quad \mathrm{Log}3\,, \quad \mathrm{Log}2\,.$

Weil's proof

The proof of RH rests on two results

- (A) Positivity : $\text{Trace}(Z \star Z') > 0$ unless Z is a trivial class.

- (B) Explicit Formula

$$\#\{C(\mathbb{F}_{q^j})\} = \sum (-1)^k \, \mathrm{Tr}(\mathrm{Fr}^{*j} | H^k_{\mathrm{et}}(\bar{C}, \mathbb{Q}_\ell))$$

The role of the positivity condition (A) in Weil's proof is contained in the following :

The following two conditions are equivalent :

- **All L functions with Grössencharakter on K satisfy the Riemann Hypothesis.**

- $\mathrm{Trace}_{H^1}(f \star f^\sharp) \geq 0$ **for all** $f \in \mathcal{S}(C_K)$**.**

$$f \to f^\sharp \,, \quad f^\sharp(g) = |\,g|^{-1}\, \bar{f}(g^{-1})$$

Weil's proof : Correspondences

$Z\ :\ C \to C,\ P \to Z(P)$

$$U \sim V \Leftrightarrow U - V = (f)$$

$$Z = Z_1 \star Z_2\,, \quad Z_1 \star Z_2(P) = Z_1(Z_2(P))$$

$$Z' = \sigma(Z)$$

$$d(Z) = Z \bullet (P \times C)\,, \quad d^{'}(Z) = Z \bullet (C \times P)$$

Weil defines the *Trace* of a correspondence as follows

$$\text{Trace}(Z) = d(Z) + d^{'}(Z) - Z \bullet \Delta$$

where Δ is the identity correspondence and $\bullet$ is the intersection number.

Proof of positivity (A)

In any (correspondence class)/(trivial ones) one finds a representative Z such that

$$Z > 0\,, \quad d(Z) = g$$

Writing $Z(P) = Q_1 + \cdots + Q_g$, $Z \star Z'(P)$ is the locus of $\sum Q_i \times Q_j$,

$$Z \star Z' = d^{'}(Z)\,\Delta + Y$$

$$Y \bullet \Delta \leq (4\,g - 4)\,d^{'}(Z)\,,$$

$$K(P) = \det\{f_i(Q_j)\}^2$$

$$\Delta \bullet \Delta = 2 - 2\,g$$

$$\mathrm{Trace}(Z \star Z') = 2\,g\,d^{'}(Z) + (2\,g - 2)\,d^{'}(Z) - Y \bullet \Delta$$

$$\geq (4\,g - 2)\,d^{'}(Z) - (4\,g - 4)\,d^{'}(Z) = 2\,d^{'}(Z) \geq 0$$

because $d^{'}(Z) \geq 0$ since Z is effective.

Virtual correspondences	bivariant class Γ
Degree of correspondence	Pointwise index $d(\Gamma)$
$\deg D(P) \geq g \Rightarrow\sim$ effective	$d(\Gamma) > 0 \Rightarrow \exists K, \Gamma + K$ onto
Adjusting the degree by trivial correspondences	Fubini step on the test functions
Frobenius correspondence	bivariant element $\Gamma(h)$
Lefschetz formula	bivariant Chern of $\Gamma(h)$ (localization on graph $Z(h)$)